UNIT – I
MATRICES

Introduction of Matrices:

Definition :

A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote matrices by capital letters as *A,B, C* etc

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{m \times n}$$

Remark: A matrix is not just a collection of elements but every element has assigned a definite position in a particular row and column.

Sub – Matrix: Any matrix obtained by deleting some rows or columns or both of a given matrix is called sub matrix.

Minor of a Matrix: let A be an mxn matrix. The determinant of a square sub matrix of A is called a minor of the matrix.

Note: If the order of the square sub matrix is 't' then its determinant is called a minor of order 't'.

Rank of a Matrix:

Definition:

A matrix is said to be of rank r if

i. It has at least one non-zero minor of order r and
ii. Every minor of order higher than r vanishes.

Rank of a matrix A is denoted by ρ(A).

Properties:

1) The rank of a null matrix is zero.
2) For a non-zero matrix A , $\rho(A) \geq 1$
3) The rank of every non-singular matrix of order n is n. The rank of a singular matrix of order n is < n.
4) The rank of a unit matrix of order n is n.
5) The rank of an $m \times n$ matrix $\leq \min(m, n)$.

6) The rank of a matrix every element of which is unity is one
7) Equivalent matrices have the same order and same rank because elementary transformation do not alter its order and rank.
8) Rank of a matrix is unique.
9) Every matrix will have a rank

Elementary Transformations on a Matrix:

i) Interchange of i^{th} row and j^{th} row is denoted by $R_i \leftrightarrow R_j$

ii) If i^{th} row is multiplied with k then it is denoted by $R_i \to k\, R_i$

iii) If all the elements of i^{th} row are multiplied with k and added to the corresponding elements of j^{th} row then if is denoted by $R_i \to R_i + kR_j$.

Note: 1. The corresponding column transformations will be denoted by writing 'c'

$$i.e\; c_i \leftrightarrow c_j,\quad c_i \leftrightarrow k\,c_j\;,\quad c_i \to c_i + kc_j$$

2. The elementary operations on a matrix do not change its rank.

Equivalence of Matrices: If B is obtained from A after a finite chain of elementary transformations then B is said to be equivalent to A. It is denoted as B~A.

Different methods to find the rank of a matrix:

Method 1:

Echelon form: A matrix is said to be in Echelon form if

1) Zero rows, if any, are below any non-zero row

2) The number of zeros before the first non-zero elements in a row is less than the number of such zeros in the next rows.

Ex: The rank of matrix which is in Echelon form $\begin{bmatrix} 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 9 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is 3 since the no. of non-zero rows is 3

Note:1. Apply only row operations while reducing the matrix to echelon form

Problem: Find the rank of the matrix A = $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ by reducing into echelon form

Sol: $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

R_3-3R_1, R_4-R_1

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

R_3-R_2, R_4-R_2

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in echelon form

Rank = no. of non zero rows = 2

Method 2:

<u>Normal Form</u>: Every m×n matrix of rank r can be reduced to the form of $I_r, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, [I_r \quad 0], \begin{bmatrix} I_r \\ 0 \end{bmatrix}$ by a finite chain of elementary row or column operations where I_r is the Identity matrix of matrix of order r.

Normal form another name is "canonical form"

Problem: Find the rank of matrix $A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ by reducing it to canonical form

Sol: Given matrix A = $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 4 & 2 & 0 & 2 \\ 2 & -2 & 0 & 6 \\ 1 & -2 & 1 & 2 \end{bmatrix} R_2-4R_1, R_3-2R_1, R_4-R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} C_2+C_1, C_4-3C$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{bmatrix} R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \end{bmatrix} (-1)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & -10 \end{bmatrix} R_4-6R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -16 \end{bmatrix} C_3+C_2, C_4-C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & -16 \end{bmatrix} \frac{1}{6}C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -16 \end{bmatrix} C_4+16C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} I_3 & O \\ O & O \end{bmatrix}$$

The above matrix is in normal form and rank is 3.

Finding the Inverse of a Nonsingular Matrix using Row/Column Transformations(Gauss-Jordan Method):

EXAMPLE: ***Find the inverse of the matrix*** $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ ***using the Gauss-Jordan method.***

Solution: Consider the matrix $\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$. ***A sequence of steps in the Gauss-Jordan method are:***

1. $$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1(1/2)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\overrightarrow{R_{31}(-1)}]{\overrightarrow{R_{21}(-1)}} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix} \xrightarrow{R_2(2/3)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

4. $$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix} \xrightarrow{R_{32}(-1/2)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix}$$

5. $$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix} \xrightarrow{R_3(3/4)} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

6. $$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & \frac{-1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{bmatrix} \begin{matrix} \xrightarrow{R_{23}(-1/3)} \\ \xrightarrow{R_{13}(-1/2)} \end{matrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{5}{8} & \frac{1}{8} & \frac{-3}{8} \\ 0 & 1 & 0 & \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{bmatrix}$$

7. $$\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{5}{8} & \frac{1}{8} & \frac{-3}{8} \\ 0 & 1 & 0 & \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{bmatrix} \xrightarrow{R_{12}(-1/2)} \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} \\ 0 & 1 & 0 & \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{bmatrix}.$$

8. Thus, the inverse of the given matrix is $$\begin{bmatrix} 3/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & 3/4 \end{bmatrix}.$$

EXERCISE : ***Find the inverse of the following matrices using the Gauss-Jordan method.***

$$(i) \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}, \quad (ii) \begin{bmatrix} 1 & 3 & 3 \\ 2 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}, \quad (iii) \begin{bmatrix} 2 & -1 & 3 \\ -1 & 3 & -2 \\ 2 & 4 & 1 \end{bmatrix}.$$

Elementary matrix:

A matrix obtained from a unit matrix by a single elementary transformation.

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Linear Equation: An Equation is of the form

$a_1x_1+a_2x_2+a_3x_3+\ldots..+\ldots.a_nx_n = b$ where $x_1,x_2,----,x_n$ are unknown and $a_1,a_2,\ldots a_n$, b are constants is called a linear equation in 'n' unknowns.

Consistency of System of Linear equations (Homogeneous and Non Homogeneous) Using Rank of the Matrix:

A System of m linear algebraic equations in n unknowns $x_1, x_2, x_3, \ldots\ldots x_n$ is a set of equations of the form

$$\left.\begin{array}{llll} a_{11}x_1 + & a_{12}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots + & a_{1n}x_n = b_1 \\ a_{21}x_1 + & a_{22}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots + & a_{2n}x_n = b_2 \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{m1}x_1 + & a_{m2}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots +. & a_{mn}x_n = b_m \end{array}\right\} \to (1)$$

The numbers a_{ij}'s are known as coefficients and b_i are known as constants of the system (1) can be expressed as $\sum_{j=i}^{n} a_{ij} x_j$ = b_i ,i = 1,2,….m

Non homogenous System : When at least one b_i is nonzero .

Homogenous System: If $b_i = 0$ for i = 1,2, ….m (all R.H.S constants are zero) Solution of system (1) is set of numbers $x_1, x_2, x_3, \ldots\ldots x_n$ which satisfy simultaneously all the equations of the system (1)

Trivial Solution is a solution where all x_i are zero i.e $x_1 = x_2 = \ldots\ldots = x_n = 0$

The set of equations can be written in matrix form as AX = B $\to$ (2)

Where A = $\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots\ldots & & \\ a_{m1} & a_{m2} & \ldots\ldots\ldots & a_{mn} \end{bmatrix}$ is called the coefficient matrix

X = $\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ is the set of unknowns B = $\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_m \end{bmatrix}$ is a column matrix of constants

<u>**Consistent**</u> : A system of equations is said to be consistent if (1) has at least one solution.

Inconsistent if system has no solution at all

Augmented matrix [A B] of system (1) is obtained by augmenting A by the column B

Matrix equation for the homogenous system of equations is AX = 0 ….(3)

It is always consistent.

If X_1, X_2 are two solutions of equation (3) then their linear combination $k_1x_1 + k_2x_2$ where k_1 & k_2 are any arbitrary numbers, is also solution of (3) .The no. of L.I solutions of m homogenous linear equations in n variables , AX = 0 , is (n – r) where r is the rank of the matrix A.

Nature of solution:

non-homogeneous with m equations and n unknowns

The system of equations AX=B is said to be

i. consistent and unique solution if rank of A = rank of [A|B] = n i.e.,r = n

   Where r is the rank of A and n is the no. of unknowns.

ii. Consistent and an infinite no. of solutions if rank of A= rank of [A|B] < n i.e., r < n. In this case we have to give arbitrary values to n-r variables and the remaining variables can be expressed in terms of these arbitrary values.

iii. Inconsistent if rank of A ≠ rank of [A|B]

Note: Method of finding the rank of A and [A|B]:

Reduce the augmented matrix [A:B] to Echelon form by elementary row transformations.

Problem: Show that the equations x+y+z = 4, 2x+5y-2z =3, x+7y-7z =5 are not consistent.

Sol: write given equations is of the form AX = B

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \\ 1 & 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

Consider augmented matrix $\left[A/B\right] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 5 & -2 & 3 \\ 1 & 7 & -7 & 5 \end{bmatrix}$

$$Applying\ R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 6 & -8 & 1 \end{bmatrix}$$

$$Applying\ R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 3 & -4 & -5 \\ 0 & 0 & 0 & 11 \end{bmatrix}$$

Thus $\rho(A) = 2$ *and* $\rho[A/B] = 3$

Therefore $\rho(A) \neq \rho[A/B]$

Hence the given system is an inconsistent.

<u>Homogeneous linear equations:</u>

Consider the system of m homogeneous equations in n unknowns

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{mn}x_n = 0 \end{array}\right. \qquad (1)$$

(1) can be written as AX=O

Where A is the coefficient matrix formed by A = $\begin{bmatrix} a_{11} & a_{12} & - & - & a_{1n} \\ a_{21} & a_{22} & - & - & a_{2n} \\ - & - & - & - & - \\ - & - & - & - & - \\ a_{m1} & a_{m2} & - & - & a_{mn} \end{bmatrix}$

X= $\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ and B= $\begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$

Consistency: The matrix A and [A|B] are same. So rank of A = rank of [A|B]

Therefore the system (1) is always consistent.

Nature of solution:

Trivial solution: Obviously $x_1=x_2=x_3=$ -------- $=x_n=0$ is always a solution of the given system and this solution is called trivial solution.

Therefore trivial solution or zero solution always exists.

Non-Trivial solution: Let r be the rank of the matrix A and n be the no. of unknowns.

Case-I: If r=n, the equations AX=O will have n-n i.e., no linearly independent solutions. In this case, the zero solution will be the only solution.

Case-II: If r<n, we shall have n-r linearly independent solutions. Any linear combination of these n-r solutions will also be a solution of AX=O.

Case-III: If m<n then $r\le m<n$. Thus in this case n-r > 0.

Therefore when the no. of equations < No. of unknowns, the equations will have an infinite no. of solutions.

Note: The system Ax =0 possesses a non-zero solution if and only if A is a singular matrix.

Ex: Show that the only real number λ for which the system

$X+2y+3z = \lambda x,\ 3x+y+2z = \lambda y,\ 2x+3y+z = \lambda z$ has non-zero solution is 6 and solve them when $\lambda=6$.

Sol: Given system of equations can be expressed as AX=O

Where $A=\begin{bmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{bmatrix}$; $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here the no. of variables = n = 3.

The given system of equations possesses a non-zero (non-zero) solution, if rank of A < number of unknowns i.e., rank of A < 3.

For this we must have detA = 0

$$\therefore \quad \begin{vmatrix} 1-\lambda & 2 & 3 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix} = 0$$

i.e., $\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix}=0$ $\quad R_1 \to R_1+R_2+R_3$

i.e., $(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1-\lambda & 2 \\ 2 & 3 & 1-\lambda \end{vmatrix}=0$

i.e., $(6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix}=0$ $\quad C_2 \to C_2-C_1,\ C_3 \to C_3-C_1$

i.e., $(6-\lambda)(\lambda^2+3\lambda+3) = 0$

i.e., $\lambda = 6$ is the only real value and other values are complex.

When $\lambda = 6$, the given system becomes

$$\begin{bmatrix} -5 & 2 & 3 \\ 3 & -5 & 2 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 19 & -19 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_2 \to 5R_2+3R_1,\ R_3 \to 5R_3+2R_1$$

$$\Rightarrow \begin{bmatrix} -5 & 2 & 3 \\ 0 & -19 & 19 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad R_3 \to R_3+R_2$$

$\Rightarrow$ $-5x+2y+3z = 0$ and $-19y+19z = 0$

$\Rightarrow$ $y=z$

Since rank of A < No. of unknowns i.e., $r < n$ $(2 < 3)$

Therefore, the given system has infinite no. of non-trivial solutions.

Let z=k $\Rightarrow$ y=k and -5x+2k+3k = 0 $\Rightarrow$ x=k

$\therefore$ x=k, y=k and z=k is the solutions

GAUSSIAN ELIMINATION METHOD: Consider the system of linear equation s

$$\begin{aligned} a_1x_1+b_1x_2+c_1x_3&=d_1\\ a_2x_1+b_2x_2+c_2x_3&=d_2\\ a_3x_1+b_3x_2+c_3x_3&=d_3 \end{aligned}$$

The augmented matrix is [A:B]= $\begin{bmatrix} a1 & b1 & c1 & d1 \\ a2 & b2 & c2 & d2 \\ a3 & b3 & c3 & d3 \end{bmatrix}$

After performing row operations or column operations we get

$$[A:B] = \begin{bmatrix} a1 & b1 & c1 & d1 \\ 0 & b2 & c2 & d2 \\ 0 & 0 & c3 & d3 \end{bmatrix}$$

Ex: solve the equations

2x1 + x2 + x3 = 10

3x1 + 2x2 +3x3 = 18

X1 + 4x2 + 9x3 = 16 using gauss elimination method

Sol: The augmented matrix of the given matrix is [A B] = $\begin{bmatrix} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{bmatrix}$

[A B] ~ $\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 7/2 & 17/4 & 11 \end{bmatrix}$ $R_2 \to 2R_3 - R_1$

$$[A\ B] \sim \begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & 1/2 & 3/2 & 3 \\ 0 & 0 & -2 & -10 \end{bmatrix} \quad R_3 \to R_3 - 7R_2$$

This augmented matrix corresponds to the following upper triangular system. By backward substitution we get

$$2x1 + x2 + x3 = 10;\ \frac{1}{2}x2 + \frac{3}{2}x3 = 3;\ -2x3 = -10$$

x3 = 5 ; x2 = -9 ; x1 = 7

Introduction of LU Decomposition :

A m×n matrix is said to have a **LU-decomposition** if there exists matrices L and U with the following properties:

(i) L is a m×n lower triangular matrix with all diagonal entries being 1.

(ii) U is a m×n matrix in some echelon form.

(iii) $A = LU$.

Procedure to solve by LU Decomposition:

Suppose we want to solve a m×n system $AX = \boldsymbol{b}$.

If we can find a LU-decomposition for A , then to solve $AX = \boldsymbol{b}$, it is enough to solve the systems

$$\left.\begin{aligned} LY &= b \\ UX &= Y \end{aligned}\right\}.$$

Thus the system LY = b can be solved by the method of forward substitution and the system $UX = Y$

can be solved by the method of backward substitution. To illustrate, we give some examples

It turns out that we need only consider lower triangular matrices L that have 1's down the diagonal.

Here is an example, let $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}$ = LU where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Multiplying out LU **and** setting the answer equal to A gives

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} & l \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$
. Now we have to use this to find the entries in L and U.

Fortunately this is not nearly as hard as it might at first seem. We begin by running along the top row to see that $u_{11} = 1$, $u_{12} =5$, $u_{13} = 1$. Now consider the second row $l_{21}u_{11} = 2 \therefore l_{21} \times 1=2 \therefore l_{21} = 2$, $l_{21}u_{12} + u_{22} = 1 \therefore 2 \times 5 +u_{22} = 1 \therefore u_{22} =-9$, $l_{21}u_{13} + u_{23} = 3 \therefore 2 \times 1 + u_{23} = 3 \therefore u_{23} =1$.

Now we solve the system LY=B i.e.,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 14/9 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 9 \\ 12 \\ 16 \end{bmatrix}$$
by forward substitution $y_1=9,y_2=-6,y_3=-5/3$

And the system UX=Y i.e.,
$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ -6 \\ -5/3 \end{bmatrix}$$
by backward substitution x=1,y=1,z=3.

A. Objective Questions

1. The rank of $\boldsymbol{I_3}$ = ___________
2. The rank of $\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{3} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{pmatrix}$ is ___________
3. If the rank of a matrix is 4. Then the rank of its transpose is__________
4. The rank of a matrix in echelon form is equal to ________
5. The necessary and sufficient condition that the system of equations AX=B is consistent if_________________
6. The value of K for which the system of equations 5x+3y=12, 15x+9y=k-3 has infinitely many solution is ____________
7. The non trivial solution of system of equations $2x-3y=0$ *and* $-4x+6y=0$ *is* ______
8. The system of equations $x+y+z=3, x+2y+3z=4, x+4y+9z=6$ will have ______
9. If the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 5 \\ 2 & k & 4 \end{bmatrix}$ is 2 then k=_________

10.The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

(a) 0 (b) 1 (c) 2 (d) 3

11. If 5 non homogeneous equations are given with 4 unknowns. The system of equations AX=B consistent if

(a) The rank of A=4 (b) the rank of A is 3
(c) the rank of A <4 (d) the rank of A is 5

12.If the system of equations $x - 3y - 8z = 0, 3x + y - \lambda z = 0, 2x + 3y + 6z = 0$ possess a nontrivial solution then $\lambda =$

(a) 2 (b) $\frac{-4}{9}$ (c) 6 (d) 8

13. Every square matrix can be written as a product of lower and upper triangular matrices if

(a)atleast one principal minor is zero (b) all principal minors are non-zero
(c) all principal minors are zero (d) atleast one principal minor is non- zero

14. Consider two statements:

i. P: Every matrix has rank
ii. Q: Rank of a matrix is not unique

(a) Both P and Q are false (b) Both P and Q are true
(c) P is true and Q is false (d) P is false and Q is true

15. Which of the following statement is correct

a. Rank of a Non-zero matrix is Zero

b.Rank of a rectangular matrix of order mxn is m when m > n

c.Rank of a rectangular matrix of order mxn is m when m < n

d.Rank of a square matrix of order nxn is n+1.

16. Rank of a non singular matrix of order m is

a. m b. n c. 0 d. not defined

17. Rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is

a. 1 b. 2 c. 3 d. 4

8. Find the values of k_1 and $k_2$for which the non-homogeneous linear

system,

$3x - 2y + z = k_2$; $5x - 8y + 9z = 3$; $2x + y + k_1 z = -1$ has no solution

a) $k_1 = -3$, $k_2 = 1/3$ b) $k_1 = 3$, $k_2 \neq 1/3$

c) $k_1 = -3$, $k_2 \neq 1/3$ d) $k_1 = 3$, $k_2 = 1/3$

19. The equations $x + 4y + 8z = 16$, $3x + 2y + 4z = 12$ and $4x + y + 2z = 10$

have

a) only one solution b) two solutions

c) infinitely many solutions d) no solutions

B. Subjective Questions :

1. Determine the rank of matrix by reducing to echelon form

i) $A = \begin{bmatrix} 1 & -1 & -1 & 2 \\ 4 & 2 & 2 & -1 \\ 2 & 2 & 0 & -2 \end{bmatrix}$ ii) $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

iii) $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ $vi)\ A = \begin{bmatrix} 3 & -1 & 2 & 1 \\ 1 & 4 & 6 & 1 \\ 7 & -11 & -6 & 1 \\ 7 & 2 & 12 & 3 \end{bmatrix}$

2. Find the rank of the following matrices by reducing them into Normal form.

a) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 10 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

3. Find the rank of the following matrices by reducing them into Canonical form

$\begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$, $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

4. Test for the consistency and solve the following equations: 2x-3y+7z =5; 3x+y-2z =13; 2x +19y -47z =32
5. Investigate for what values of a and b the simultaneous equations x +a y + z = 3; x + 2y +2z = b; x +5y +3z =9 have
   a) no solution b) a unique solution c) infinitely many solutions
6. Test for consistency and solve if the equations are consistent $x+2y+2z=2, 3x-y+3z=-4, x+4y+6z=0$
7. Solve the system of equations by using LU Decomposition method 3x+2y+2z=4, 2x+3y+z=5, 3x+4y+z=7.
8. Express A = $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$ as a product of LU. .
9. Test for the consistency of following and solve the following equations: x +2y +z =3 ; 2x +3y+2z =5; 3x -5y +5z =2; 3x +9y –z =4
10. For what value of k the equations x + y + z =1;2x + y + 4z = k;4x + y + 10z=k^2 have a solution and solve them completely in each case.

(C). GATE Previous Paper Questions

1. The system of linear equations $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 5` \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}=\begin{bmatrix} 5 \\ -4 \\ 14 \end{bmatrix}$ has

   (GATE 2014)

   a) A unique solution b) infinitely many solutions
   b) No solution d) exactly two solutions
2. The system of equations x + y + z =6, x + 4y + 6z = 20, x + 4y + λz = u (GATE 2011)

   has no solution for values of λ and μ given by

   a) $\lambda=6, \mu=20$ b) $\lambda=6, \mu\neq 20$
   c) $\lambda\neq 6, \mu=20$ d) $\lambda\neq 6, \mu\neq 20$
3. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is (GATE 2006)

   a) 0 b) 1 c) 2 d) 3

4. The determinant of a matrix A is 5 and the determinant of matrix B is 40 the determinant of matrix AB is______________ (GATE2014)
5. Consider the following system of equations 3x + 2y = 1, 4x + 7z = 1, x + y + z =3, x-2y+7z=0 The number of solution for this system is (GATE 2014)
6. The following system of equations $x_1+x_2+2x_3=1$, $x_1+2x_2+3x_3=2$, $x_1+4x_2+\alpha x_3=4$ has α unique solution the only possible values of α are (GATE2008)

   a) 0 b) either 0 or 1 c) one of 0,1, or -1 d) any real number
7. Consider the following system of equations in three variables x_1,x_2 and x_3 $2x_1-x_2+3x_3=1$, $3x_1+2x_2+5x_3=2$, $-x_1+4x_2+x_3=3$ then The system of equations has (GATE 2005)

a) No Solutions b) More than one but a finite number of solutions

c) Unique solutions d) All infinite number of solutions

8. How many solutions does the following system of linear equations have –x + 5y = -1, x + 3y =3 , x – y =2 (GATE 2013)

   a) Infinitely many b) Two distinct solutions

   c) Unique d) None
9. For matrices of same dimension M and N and a scalar C which of these properties does not always hold (GATE 2014)

   a) $(M^T)^T = M$ b) $(CM)^T = CM^T$

   c) $(M+N)^T = M^T + N^T$ d) MN = NM
10. In the LU decomposition of the matrix $\begin{bmatrix} 2 & 2 \\ 4 & 9 \end{bmatrix}$, if the diagonal elements of U are both 1,then lower diagonal entry l_{22} of L is ___________. (GATE 2009)

   a) 4 b) 5 c) 6 d) 7

UNIT-II

EIGEN VALUES AND EIGEN VECTORS

Eigen values and eigen vectors of a matrix:

Consider the following 'n' homogeneous equations in 'n' unknowns as given below

$$(a_{11} - \lambda)\, x_1 + a_{12}x_2 + \text{------}\ a_{1n}x_n = 0$$

$$a_{21}x_1 + (a_{22} - \lambda)\, x2 + \text{------}\ a_{2n}x_n = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \text{------}\ (a_{nn} - \lambda)x_n = 0$$

The above system of equations in matrix notation can be written as $(A - \lambda I)\, X = 0$

Where 'λ' is a parameter.

The matrix $(A - \lambda I)$ is called 'Characteristic Matrix' and $|A - \lambda I| = 0$ is called 'Characteristic Equation' of A . i.e.,

$|A - \lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \text{------} + k_n = 0$

Where $k_1, k_2, \text{-----}, k_n$ are expressible in terms of the elements a_{ij}

Eigen Value: The roots of characteristic equation are called the characteristic roots or latent roots or eigen values.

Eigen Vector: If λ is a characteristic root of a matrix then a non-zero vector X such that $AX = \lambda X$ is called a characteristic vector or Eigen vector of A corresponding to the characteristic root λ.

Note: (i) Eigen vector must be a non-zero vector

(ii) Eigen vector corresponding to a eigen value need not be unique

PROPERTIES OF THE EIGEN VALUES:

- The sum of the Eigen values of the square matrix is equal to its trace and product of the Eigen values is equal to its determinant.

- If λ is an eigen value of A corresponding to the eigen vector X then λ^n is the eigen value of the matrix A^n corresponding to the eigen vector X .
- If $\lambda_1, \lambda_2, \lambda_3, ----, \lambda_n$. are the latent roots of A then A^3 has the latent roots as $\lambda^3{}_1, \lambda^3{}_2, \lambda^3{}_3, ----, \lambda^3{}_n$.
- A square matrix A and its transpose A^T have the same eigen values.
- If A and B are n rowed square matrix and if A is invertible then $A^{-1}B$ and BA^{-1} have the same eigen values.
- If $\lambda_1, \lambda_2, \lambda_3, ----, \lambda_n$. are the eigen values of a matrix A then $k\lambda_1, k\lambda_2, k\lambda_3, ----, k\lambda_n$. are the eigen values of the matrix KA where K is a non-zero scalar.
- If λ is an Eigen value of the matrix A then $\lambda+k$ is an Eigen value of the matrix A+KI.
- If λ is the Eigen value of A then λ-K are the eigen values of the matrix A-KI.
- If $\lambda_1, \lambda_2, \lambda_3, ----, \lambda_n$. are the eigen values of a matrix A then $(\lambda_1 - \lambda)^2, (\lambda_2 - \lambda)^2, \ldots\ldots\ldots\ldots (\lambda n - \lambda)^2$ are the eigern values of the matrix $(A- \lambda I)^2$.
- If λ is an Eigen value of a non-singular matrix A then λ^{-1} is an Eigen value of the matrix A^{-1} corresponding to the eigen vector X .
- If λ is an Eigen value of a non-singular matrix A then $|A|/\lambda$ is an eigen value of the matrix adjA.
- If λ is an Eigen value of a non-singular matrix A then $1/\lambda$ is an eigen value of A^{-1}.
- If λ is an Eigen value of a non-singular matrix A then the eigen value of $B = a_0A^2+a_1A+a_2I$ is $a_0\lambda^2+a_1\lambda+a_2$.
- The eigen values of a triangular matrix are just diagonal elements of the matrix.
- If A and B are non-singular matrices of same order,then AB and BA have the same eigen values.
- Suppose A and P are square matrices of order n such that P is non-singular,then A and $P^{-1}AP$ have the same eigen values.
- The eigen values of real symmetric matrix are real.

- For a real symmetric matrix ,the eigen vectors corresponding to two distinct eigen values are orthogonal.

- The two eigen vectors corresponding to two different eigen values are linearly independent.

Finding Eigen vectors:

Method1:

Case(i): Eigen values are distinct $\lambda_1 \neq \lambda_2 \neq \lambda_3$ (suppose the matrix A of order 3)

Corresponding to the eigen value λ_1, the eigen vector $X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ can be obtained from the matrix equation $(A- \lambda_1)X_1=O$ and by expanding it we get three homogeneous linearly independent equations are obtained and solving any two equations for x_1,x_2,x_3 the eigen vector

$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ can be obtained. Similarly, the remaining Eigen vectors X_2, X_3 can be obtained corresponding to the Eigen values λ_2 and λ_3.

Case(ii): Finding Linearly Independent Eigen vectors of a matrix when the Eigen values of the matrix are repeated $(\lambda_1=\lambda_2)$

The matrix equation $(A- \lambda I)\ X=O$ gives three equations which represent a single independent equation of the form.

$a_1x_1+a_2x_2+a_3x_3 = 0$

We have to choose two unknowns as k_1,k_2.

So we can get two linearly independent Eigen vectors X_1 and X_2

Method2: (Rank method) in the matrix equation $(A-\lambda I)\ X=O$, reduce the coefficient matrix to Echelon form, the rank of the coefficient matrix is less than the number of unknowns. So give arbitrary constants to (n-r) variables and solve as in case of homogeneous equations.

Example 1: Find the Eigen values and corresponding Eigen vectors of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol: The characteristic equation of the matrix A is $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad \lambda^3 - 10\lambda^2 + 20\lambda - 32 = 0 \Rightarrow \quad (\lambda-2)(\lambda-2)(\lambda-8) = 0 \Rightarrow \quad \lambda = 2, 2, 8$

The Eigen values of A are 2, 2 and 8.

Let the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to the Eigen value λ is given by the non-zero solution of the equation $(A-\lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\lambda=8$, then the Eigen vector X_1 is given by $(A-8I)X_1 = O$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -2x_1-2x_2+2x_3 = 0, -2x_1-5x_2-x_3 = 0, 2x_1-x_2-5x_3 = 0$

Solving any two of the equations, we get $\Rightarrow \dfrac{x_1}{2} = \dfrac{x_2}{-1} = \dfrac{x_3}{1} = k$ (say)

$\Rightarrow x_1=2k, x_2=-k, x_3=k$ (k is arbitrary) $\Rightarrow \quad X_1 = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} \Rightarrow \quad X_1 = k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

The eigen vector corresponding to $\lambda_1=8$ is $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

If $\lambda=2$, then the Eigen vector X_2 is given by $(A-2I)X_2 = O$

$$\Rightarrow \begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $4x_1-2x_2+2x_3 = 0$, $-2x_1+x_2-x_3 = 0$, $2x_1-x_2+x_3 = 0$ $\Rightarrow$ $2x_1-x_2+x_3 = 0$

Let $x_2 = k_2$, $x_3 = k_1$, $2x_1 = k_2-k_1$ $\therefore$ $X_2= \begin{bmatrix} \frac{k_2-k_1}{2} \\ k_2 \\ k_1 \end{bmatrix} =2 \begin{bmatrix} \frac{k_2-k_1}{} \\ 2k_2 \\ 2\,k_1 \end{bmatrix} = 2k_1 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} +2k_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\therefore$ Eigen vectors corresponding to $\lambda= 2$ are $X_2 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ $X_3= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$\therefore$ Eigen vectors of A are $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Cayley-Hamilton theorem:

- *Every square matrix satisfies its own characteristic equation.*

Remark: (i) Determination of A^{-1} using Cayley-Hamilton theorem

Let A be n-rowed square matrix.By Cayley-Hamilton theorem ,A satisfies its own characteristic equation. i.e $(-1)^n[A^n+a_1A^{n-1}+a_2A^{n-2}+\dots\dots+a_nI] = 0$

$\Rightarrow A^n +a_1A^{n-1}+a_2A^{n-2}+\dots\dots+a_nI = 0$ ------ (1)

$\Rightarrow A^{-1}[\ A^n +a_1A^{n-1}+a_2A^{n-2}+\dots\dots+a_nI] = 0$

If A is a non-singular ,then we have $a_nA^{-1} = -A^{n-1}-a_1A^{n-2}-\dots\dots-a_{n-1}I$

$\Rightarrow A^{-1} = \left(\frac{-1}{a_n}\right)[A^{n-1}+a_1A^{n-2}-\dots\dots+a_{n-1}I]$

Remark: (ii) Determination of powers of **A** using Cayley-Hamilton theorem

Multiplying equation (1) with A, we get $A^{n+1} +a_1A^n+a_2A^{n-1}+\dots\dots+a_nA = 0$

$\Rightarrow A^{n+1} = -[a_1A^n+a_2A^{n-1}+\dots\dots+a_nA]$

Example 1: If A = $\begin{bmatrix}1 & 2 & -1\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$ verify Cayley-Hamilton theorem .Find A^4 and A^{-1} using Cayley-Hamilton theorem**.**

Solution: Given matrix is $A = \begin{bmatrix}1 & 2 & -1\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$

Characteristic equation is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix}1-\lambda & 2 & -1\\ 2 & 1-\lambda & -2\\ 2 & -2 & 1-\lambda\end{vmatrix} = 0 \Rightarrow (1-\lambda)(\lambda^2-2\lambda-3)-2(6-2\lambda)-1(-6+2\lambda) = 0$$

$\Rightarrow -\lambda^3 + 3\lambda^2 + 3\lambda - 9 = 0 \Rightarrow \lambda^3 - 3\lambda^2 - 3\lambda + 9 = 0$ ------(i)

$$A = \begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-1}\\ \mathbf{2} & \mathbf{1} & \mathbf{-2}\\ \mathbf{2} & \mathbf{-2} & \mathbf{1}\end{bmatrix}$$

$$A^2 = \begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-1}\\ \mathbf{2} & \mathbf{1} & \mathbf{-2}\\ \mathbf{2} & \mathbf{-2} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-1}\\ \mathbf{2} & \mathbf{1} & \mathbf{-2}\\ \mathbf{2} & \mathbf{-2} & \mathbf{1}\end{bmatrix} = \begin{bmatrix}\mathbf{3} & \mathbf{6} & \mathbf{-6}\\ \mathbf{0} & \mathbf{9} & \mathbf{-6}\\ \mathbf{0} & \mathbf{0} & \mathbf{3}\end{bmatrix}$$

$$A^3 = \begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-1}\\ \mathbf{2} & \mathbf{1} & \mathbf{-2}\\ \mathbf{2} & \mathbf{-2} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\mathbf{3} & \mathbf{6} & \mathbf{-6}\\ \mathbf{0} & \mathbf{9} & \mathbf{-6}\\ \mathbf{0} & \mathbf{0} & \mathbf{3}\end{bmatrix} = \begin{bmatrix}\mathbf{3} & \mathbf{24} & \mathbf{-21}\\ \mathbf{6} & \mathbf{21} & \mathbf{-24}\\ \mathbf{6} & \mathbf{-6} & \mathbf{3}\end{bmatrix}$$

Consider $A^3-3A^2-3A+9I = \begin{bmatrix}\mathbf{3} & \mathbf{24} & \mathbf{-21}\\ \mathbf{6} & \mathbf{21} & \mathbf{-24}\\ \mathbf{6} & \mathbf{-6} & \mathbf{3}\end{bmatrix} -3\begin{bmatrix}\mathbf{3} & \mathbf{6} & \mathbf{-6}\\ \mathbf{0} & \mathbf{9} & \mathbf{-6}\\ \mathbf{0} & \mathbf{0} & \mathbf{3}\end{bmatrix} -3\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{-1}\\ \mathbf{2} & \mathbf{1} & \mathbf{-2}\\ \mathbf{2} & \mathbf{-2} & \mathbf{1}\end{bmatrix}$

$+9\begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix} = O$

$A^3-3A^2-3A+9I = O$ -------- (ii)

Matrix A satisfies its own characteristic equation

Cayley-Hamilton theorem is verified by A

To find A^{-1} :Multiplying equation (ii) with A^{-1} on both sides

$A^{-1}[\ A^3-3A^2-3A+9I] = A^{-1}(\ O) \Rightarrow A^2-3A-3I+9A^{-1} = 0$

$$\Rightarrow 9A^{-1} = 3A+3I- A^2 = 3\begin{bmatrix}1 & 2 & -1\\2 & 1 & -2\\2 & -2 & 1\end{bmatrix}+3\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}-\begin{bmatrix}3 & 6 & -6\\0 & 9 & -6\\0 & 0 & 3\end{bmatrix}=\begin{bmatrix}3 & 0 & 3\\6 & -3 & 0\\6 & -6 & 3\end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{9}\begin{bmatrix}3 & 0 & 3\\6 & -3 & 0\\6 & -6 & 3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & 0 & 1\\2 & -1 & 0\\2 & -2 & 1\end{bmatrix}$$

To find A^4 : Multiplying equation (ii) with A on both sides

$$A[\ A^3-3A^2-3A+9I] = A\,(O) \Rightarrow A^4-3A^3-3A^2+9A = O$$

$$\Rightarrow A^4= 3A^3+3A^2-9A = 3\begin{bmatrix}3 & 24 & -21\\6 & 21 & -24\\6 & -6 & 3\end{bmatrix}+3\begin{bmatrix}3 & 6 & -6\\0 & 9 & -6\\0 & 0 & 3\end{bmatrix}-9\begin{bmatrix}1 & 2 & -1\\2 & 1 & -2\\2 & -2 & 1\end{bmatrix}=\begin{bmatrix}9 & 72 & -72\\0 & 81 & -72\\0 & 0 & 9\end{bmatrix}$$

<u>DIAGONALIZATION OF THE MATRIX:</u>

If a square matrix A of order n has linearly independent Eigen vectors $X_1 X_2 X_3 \ldots\ldots X_n$

Corresponding to the n Eigen values $\lambda_1\lambda_2\lambda_3\ldots\ldots\lambda_n$ respectively then a matrix P can be found such that p^{-1}AP=D (DIAGONAL MATRIX)

Let $X_1 X_2 X_3 \ldots\ldots X_n$ be the Eigen vector

$\lambda_1\lambda_2\lambda_3\ldots\ldots\lambda_n$ be eigen values

Define P=($X_1 X_2 X_3 \ldots\ldots X_n$) then

AP=PD

p^{-1}AP= p^{-1} PD

p^{-1} AP=D

<u>MODAL AND SPECTRAL MATRIX:</u>

The matrix P in the above result which diagonals the square matrix A is called the modal matrix of A and the resulting diagonal matrix D is know as spectral matrix.

Calculation of powers of Matrix:

Let A be the square matrix and

Let P a non-singular matrix such that D=P-1AP

D2=P-1A2P

Similarly D3=P-1A3P Like for n tern we have Dn=P-1AnP

Quadratic form(Upto Three Variables):

A homogeneous polynomial of second degree in n variables (x_1,x_2,-------,x_n) is called a quadratic form. The most general quadratic form is

$$Q = \begin{matrix} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n \\ a_{21}x_1 + a_{21}x_2 + \dots\dots\dots + a_{2n}x_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{nn}x_n \end{matrix}$$

Where a_{ij} are elements of a field F and we can write $Q=X^TAX = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$

The symmetric matrix is called the matrix of quadratic form

Canonical form:

let $Q=X^TAX$ be a quadratic form in n variables then there exists a real non-singular linear transformation X=PY which transforms $Q=X^TAX$ to the form $y_1^2 + y_2^2 + \dots\dots + y_n^2$ this form is called canonical form or normal form

Rank of the quadratic form : let X^TAX be a quadratic form over R . The rank r of A is called the rank of the quadratic form if (r<n) where n is order of A or det(A)=0 or A is singular . The quadratic form is called singular or non-singular

Index: The index of the quadratic form is the number of positive square terms in the canonical form

Signature: signature of the quadratic form is the difference between the number of positive and negative square terms of be the canonical form

Definiteness: A real non-singular quadratic form Q= X^TAX (with det(A)≠0) is said to be positive definite if rank and index are equal that is r=n and s=n. and the quadratic form is called negative definite if index equal to zero that is r=n and s=0 and the quadratic form is called positive semi-definite if the rank and index are equal but less than n that is s=r<n and it is negative semi- definite if index is zero

Reduction of quadratic form to canonical form:

Method1: **Orthogonalization**

Procedure:

Step1: Write the matrix A of the given quadratic form.

Step2: Find the Eigen values λ_1, $\lambda 2$, $\lambda 3$and corresponding Eigen vectors X_1 ,X_2 and X_3 in the normalized form.

Step3: Write the modal matrix P = [X_1 X_2 X_3] formed by normalized vectors.

Step4: P being an orthogonal matrix P^{-1}=P so that P^TAP = D where D is the diagonal matrix formed by Eigen values.

Step5: The canonical form is $Y^T (P^TAP) Y = \lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2$

Step6: The orthogonal transformation is X=PY

Note: The matrix A of the quadratic form is symmetric matrix and so diagonalization is by orthogonal transformation.

Nature of a quadratic form:

When the quadratic form Q=X^TAX is reduced to a canonical form, it will contain 'r' terms if the rank of A is r.

1. Rank of A = Rank of quadratic form = r = No. of terms in the canonical form.
2. The Index = No. of positive terms in the canonical form and is denoted by p
3. Signature = the difference of the no. of positive terms and the no. of negative terms i.e., 2p-r.
4. The quadratic form is

   (i) Positive definite, if all the Eigen values of A are positive i.e., r=n and p=n.

   (ii) Positive semi definite, if at least one of the Eigen values of A is zero and other positive i.e., r<n and p=0.

   (iii) Negative definite, if all the Eigen values are negative i.e., r=n and p=0

(iv) Negative semi definite, if at least one of the Eigen values of A is zero and others are negative i.e., r<n and p=0.

(v) Indefinite, if the Eigen values of A are positive and negative.

Ex: Reduce the quadratic form $6x_1^2+3x_2^2+3x_3^2-4x_1x_2+4x_1x_3-2x_2x_3$ to canonical form by an orthogonalization. Find rank, Index and signature of quadratic form.

Sol: The matrix A of the given quadratic form is

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation of the matrix A is $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda^3 - 10\lambda^2 + 20\lambda - 32 = 0$

$\Rightarrow \quad (\lambda-2)(\lambda-2)(\lambda-8) = 0$

$\Rightarrow \quad \lambda = 2, 2, 8$

The Eigen values of A are 2,2 and 8.

Let the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to the Eigen value λ is given by the non-zero solution of the equation $(A-\lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\lambda=8$, then the Eigen vector X_1 is given by $(A-8I)X = O$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $-2x_1-2x_2+2x_3 = 0$

$-2x_1-5x_2-x_3 = 0$

$2x_1-x_2-5x_3 = 0$

Solving any two of the equations, we get

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} = k \text{ (say)}$$

$\Rightarrow$ $x_1=2k$, $x_2=-k$ and $x_3=k$

For the eigen value $\lambda=8$, the eigen vector $X_3 = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix}$

In particular k=1, $X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

If $\lambda=2$, the Eigen vector X is given by (A-2I) X=O

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $4x_1-2x_2+2x_3 = 0$

$\Rightarrow$ $-2x_1+x_2-x_3 = 0$

$\Rightarrow$ $2x_1-x_2+x_3 = 0$

$\Rightarrow$ All the above three equations represent only one independent equation

$2x_1-x_2+x_3 = 0$

Let $x_1 = 0$, then $-x_2+x_3 = 0 \Rightarrow x_2 = x_3 =1$

Let $x_2 = 0$, then $2x_1+ x_3 = 0 \Rightarrow x_1 = 1, x_3 = -2$

So corresponding Eigen value $\lambda=2$, then linearly independent Eigen vectors are given by

$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

The normalized Eigen vector of $X_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = e_1$

The normalized Eigen vector of $X_2 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ -2/\sqrt{5} \end{bmatrix} = e_2$

The normalized Eigen vector of $X_3 = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = e_3$

The modal matrix in normalized form $P = [e_1\ e_2\ e_3]$

$$= \begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{6} \\ 1/\sqrt{2} & -2/\sqrt{5} & 1/\sqrt{6} \end{bmatrix}$$

$\therefore\ P^TAP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \text{diag}\,[2\ 2\ 8]$

And the quadratic form will be reduced to the normal form $2y_1^2+2y_2^2+8\,y_3^2$ by the orthogonal transformation X = PY. i.e.,

$x_1 = \dfrac{y_2}{\sqrt{5}} + \dfrac{2y_3}{\sqrt{6}}$; $x_2 = \dfrac{y_1}{\sqrt{6}} - \dfrac{y_3}{\sqrt{6}}$; $x_3 = \dfrac{y_1}{\sqrt{2}} - \dfrac{-2y_2}{\sqrt{5}} + \dfrac{y_3}{\sqrt{6}}$

Method2: **Diagonalization**

Procedure:

Step1: Write the matrix A of the given quadratic form, A is a symmetric matrix

Step2: Write A=IAI

Step3: Reduce the matrix A to diagonal form by applying row and column transformations.

Step4: Apply same row operations on prefactor I on R.H.S and identical column operations on postfactor I on R.H.S

Step5: A=IAI will be transformed to the form $D = P^TAP$

The canonical form is given by

$$Y^T D\, Y = [y_1\ y_2\ y_3]\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\Rightarrow \quad Y^T D\, Y = \lambda_1 y_1^2+\lambda_2 y_2^2+\lambda_3 y_3^2$

Ex: Reduce the quadratic form $6x_1^2+3x_2^2+3x_3^2-4x_1x_2+4x_1x_3-2x_2x_3$ to canonical form by diagonalization. Find rank, Index and signature of quadratic form.

Sol: The matrix A of the given quadratic form is

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation of the matrix A is $|A- \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda^3 - 10\lambda^2 + 20\lambda - 32 = 0$

$\Rightarrow \quad (\lambda-2)(\lambda-2)(\lambda-8) = 0$

$\Rightarrow \quad \lambda = 2,2,8$

The Eigen values of A are 2,2 and 8.

Let the Eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of A corresponding to the Eigen value λ is given by the non-zero solution of the equation $(A-\lambda I)X = O$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\lambda=8$, then the Eigen vector X_1 is given by $(A-8I)X = O$

$$\Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $-2x_1-2x_2+2x_3 = 0$

$-2x_1-5x_2-x_3 = 0$

$2x_1-x_2-5x_3 = 0$

Solving any two of the equations ,we get

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} = k \text{ (say)}$$

$\Rightarrow$ $x_1=2k$, $x_2=-k$ and $x_3=k$

For the eigen value $\lambda=8$, the eigen vector $X_3 = \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix}$

Inparticular k=1 , $X_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

If $\lambda=2$, the Eigen vector X is given by (A-2I)X=O

$$\Rightarrow \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $4x_1-2x_2+2x_3 = 0$

$\Rightarrow$ $-2x_1+x_2-x_3 = 0$

$\Rightarrow$ $2x_1-x_2+x_3 = 0$

$\Rightarrow$ All the above three equations represent only one independent equation

$2x_1-x_2+x_3 = 0$

Let $x_1 = 0$, then $-x_2+x_3 = 0 \Rightarrow x_2 = x_3 = 1$

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So corresponding Eigen value $\lambda=2$, then linearly independent Eigen vectors are given by

$X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $X_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

The normalized Eigen vector of $X_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = e_1$

The normalized Eigen vector of $X_2 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ -2/\sqrt{5} \end{bmatrix} = e_2$

The normalized Eigen vector of $X_3 = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = e_3$

The modal matrix in normalized form $P = [e_1\ e_2\ e_3]$

$$= \begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{6} \\ 1/\sqrt{2} & -2/\sqrt{5} & 1/\sqrt{6} \end{bmatrix}$$

The matrix P is orthogonal so $P^{-1} = P^T$

Diagonalization $P^TAP = D$

$$\Rightarrow \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{6} \\ 1/\sqrt{2} & -2/\sqrt{5} & 1/\sqrt{6} \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Canonical form $Y^T(P^TAP)Y = Y^T D Y$

$$= [y_1\ y_2\ y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 + 2y_2^2 + 8y_3^2$$

The rank of quadratic form = no. of non-zero terms in the canonical form = 3

The Index P = no. of positive terms in the canonical form = 3

Signature = 2P – rank of quadratic form = 3.

Objective Questions

1. Two of the eigen values of a 3×3 matrix whose determinant equals 4 are -1 and 2 then the third eigen value of the matrix is equal to_______________________
2. The Eigen values of A= $\begin{bmatrix} 1 & 0 & -0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are ______________________________
3. If the Eigen values of A are 1,3,0 then $|A| =$ ________________
4. The Eigen values of A are (1,-1,2) then the eigen values of Adj(A) are __________________
5. If one of eigen values of A is 0 then A is ______________
6. The eigen value of adj A is ________________
7. If A is orthogonal then A^{-1}= ____________
8. Can an eigen vector be a zero vector?(yes/no)
9. The eigen values of A^2 are _____________where A=$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$
10. Can a zero value be an eigen value?(yes/no)
11. If 2,1,3 are the eigen values of A then the eigen values of B=3A+2I are ______________
12. If A is a singular matrix then _______________is an eigen value.
13. Identify the relation between geometric and algebraic multiplicity.
14. The sum of two eigen values and trace of a 3×3 matrix are equal then the value of $|A|$ is_____________________
15. Compute characteristic equation of A= $\begin{bmatrix} 3 & -2 & -8 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.
16. The matrix A has eigen values $\lambda_i \neq 0$ then A^{-1}-2I+A has eigen values ----

19.The Eigen values of A are 2,3,4 then the Eigen values of 3A are __ [ ]

(a)2,3,4 (b) $\frac{1}{2},\frac{1}{3},\frac{1}{4}$ (c) -2,3,2 (d) 6,9,12

20.If A =$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A^3 =$ [ ]

(a) $2A^2 + 5A$ (b) $4A^2 + 2A$ (c) $2A^2 + 5A$ (d) $5A^2 + 2A$

Subjetive Questions :

1. Find the eigen values and eigen vectors of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

2. Obtain the latent roots and latent vectors of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

3. Find the eigen values and eigen vectors of $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$

4. Find the characteristic values and characteristic vectors of $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

5. Verify that sum of eigen values is equalto trace of A for A= $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and find the corresponding eigen vector.

6. Verify Cayley Hamilton theorem for A= $\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ Hence find A^{-1} and A^4

7. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$. Hence find A^{-1} and A^4

8. For the matrix A = $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $3A^3+5A^2-6A+2I$.

9. For the matrix A= $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}$ Find the eigen values and eigen vectors of A^{-1}

10. Using Cayley Hamilton theorem find A^4 for the matrix A= $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

GATE Previous Paper Questions:

1. Eigen vector of the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$ is [ ]

(GATE-2004)

a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ d) $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

2. For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ the eigen value corresponding to eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is [ ]

(GATE-2006)

a) 2 b) 4 c) 6 d) 8

3. The eigen value of the matrix $\begin{bmatrix} 5 & 3 \\ 3 & -3 \end{bmatrix}$ is [ ]

(GATE-1999)

a)6 b)5 c) -3 d)-4

4. The 3 characteristic roots of A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ are [ ]

(GATE-2000)

a)2,3,3 b)1,2,2 c)1,0,0 d)0,2,3

5. The sum of the eigen values of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are [ ]

(GATE-2004)

a)5 b)7 c)9 d)18

6. Eigen values of S= $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 and 1.Eigen values of S^2=SS are[ ]

(GATE-2006)

a)1,25 b)6,4 c)5,1 d)2,10

7. One of the eigen vectors of A= $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ is [ ]

(GATE-2010)

a) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ c) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

8. The minimum and maximum eigen value of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are -2, 6.what is other eigen value?

[ ]

(GATE-2007)

a)5 b)3 c)1 d)-1

9. All the four entries of 2x2 matrix P= $\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ are non-zero and one of its eigen value is zero which of the following is true? [ ]

(GATE-2008)

a) $p_{11}p_{22}-p_{12}p_{21}=1$ b) $p_{11}p_{22}-p_{12}p_{21}=-1$

c) $p_{11}p_{22}-p_{12}p_{21}=0$ d) $p_{11}p_{22}+p_{12}p_{21}=0$

10. Eigen values and the corresponding eigen vectors of a 2x2 matrix are given by

Eigen value	**Eigenvector**
$\lambda = 8$	X= $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\mu = 4$	Y= $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then the matrix is [ ]

(GATE-2006)

a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

11. The characteristic equation of A is $t^2-t-1=0$, then [ ]

(GATE-2000)

a) A^{-1} does not exist b) A^{-1} exist but cannot be determined from the data

c) $A^{-1}=A+I$ d) $A^{-1}=A-I$

12. A particular 3x3 matrix has an eigen value -1.The matrix A+I reduces to

$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,corresponding to eigen value -1 ,all eigen vectors of A are non-zero vectors of the form [ ]

(GATE-2002)

a) $\begin{bmatrix} 2t \\ 0 \\ t \end{bmatrix}, t \in R$ b) $\begin{bmatrix} 2t \\ s \\ t \end{bmatrix} s, t \in R$ c) $\begin{bmatrix} t \\ 0 \\ -2t \end{bmatrix} t \in R$ d) $\begin{bmatrix} t \\ s \\ 2t \end{bmatrix}$ s,$t \in R$

13. By Cayley-hamilton theorem A = $\begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ satisfies the relation [ ]

(GATE-2007)

a)$A+3I+2A^2=0$ b)$A^2+2A+2I=0$ c) $(A+I)(A+2I)=0$ d)$\exp(A)=0$

14. From question (13), $A^9 =$ [ ]

a)511A+510I b)309A+104I c)154A+155I d)exp(9A)

15. The number of linearly independent eigen vectors of $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is [ ]

(GATE-2007)

a) 0 b)1 c)2 d)infinite

UNIT III

FOURIER SERIES

To introduce

- fourier series representation of a given function with period 2π (or) $2l$
- half range series representation of a given function with period π (or)l.

Syllabus:

Determination of Fourier coefficients (without proof) – Fourier series – even and odd functions – Fourier series in an arbitrary interval– Half-range sine and cosine series.

Outcomes:

Students will be able to

- expand the given function as Fourier series in the interval [c,c+ 2π]
- expand the given function as Fourier series in the interval [c,c+ $2l$]
- expand the given function as Half-range Sine [or] Cosine series in the interval [0,l].
- write the expansions of $\frac{\pi^2}{8},\frac{\pi^2}{6},\frac{\pi^2}{12},\ldots.$

Learning Material

Introduction:

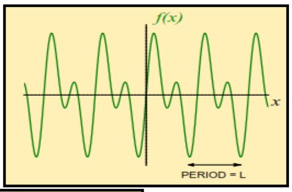
It became important to study the possibility of representation of the given function by infinite series other than power series. Since many phenomenalike ***vibration of string, the voltages and currents in electrical networks, electro- magnetic signals, and movement of pendulum are periodic in nature***.

There is a possibility of representing a periodic function as an infinite series involving sinusoidal (sin x & cosx) functions.The French physicist J.B. Fourier announced in his work on heat conduction that an arbitrary periodic function could be expanded in a series of sinusoidal functions.

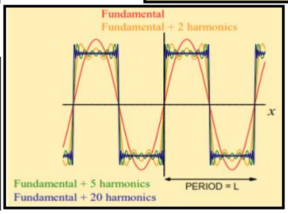
Thus, ***the aim of the theory of Fourier series is to determine the conditions under which the periodic functions can be represented as linear combinations of sine and cosine functions***.

Fourier methods give us a set of powerful tools for representing any periodic function as a sum of sines and cosines.

A graph of **periodic** function $f(x)$ that has period L exhibits the same pattern every L units along the x-axis, so that $f(x+L) = f(x)$ for every value of x. If we know what the function looks like over one complete period, we can thus sketch a graph of the function over a wider interval of x (that may contain many periods)



One can even approximate a square-wave pattern with a suitable sum that involves a fundamental sine-wave plus a combination of harmonics of this fundamental frequency. This sum is called a **Fourier series**



Existence of Fourier series:

- **Dirichlet's Conditions :**

If a function f(x) is defined in $l \le x \le l + 2\pi$, it can be expanded as a Fourier series provided the following Dirichlet's conditions are satisfied

1. **f(x) is singe valued and finite in the interval (c , c + 2π)**
2. **f(x) is piece-wise continuous with finite number of discontinuities in (c ,c + 2π).**
3. **f(x) has finite number of maxima or minima in (c ,c + 2π).**

Note:

- These conditions are not necessary but only sufficient for the existence of Fourier series.
- If f(x) satisfies Dirichlet's conditions and f(x) is defined in (c, c + 2π), then f(x) need not be periodic for the existence of Fourier series of period 2π.
- If x = a is a point of discontinuity of f(x), then the value of the Fourier series at x = a is $\frac{1}{2}[f(a+)+f(a-)]$.

Basic Formulae to Solve Integration :

Continuous Derivatives Continuous Integration

- Bracketing Method –
  [Through Examples]

$$\int x^2 . Sin\, nx\, dx = (x^2)\left(-\frac{Cos\, nx}{n}\right) - (2x)\left(-\frac{Sin\, nx}{n^2}\right) + (2)\left(\frac{Cos\, nx}{n^3}\right)$$

- $\int x.Cos\,nx\,dx = (x)\left(\frac{Sin\,nx}{n}\right) - (1)\left(-\frac{Cos\,nx}{n^2}\right)$

- $\int x^2.Cos\frac{n\pi x}{L}dx = (x^2)\left(\frac{Sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right) - (2x)\left(-\frac{Cos\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right) + (2)\left(\frac{Cos\frac{n\pi x}{L}}{\frac{n^3\pi^3}{L^3}}\right)$

❖ Spl. Formulae to Remember –

- $\boxed{\int e^{ax}.Sin\,bx\,dx = \frac{e^{ax}}{a^2+b^2}[a.Sin\,bx - b.Cosbx]}$ $\boxed{\int e^{ax}.Cosbx\,dx = \frac{e^{ax}}{a^2+b^2}[a.Cosbx + b.Sinbx]}$

- $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ [Here f(x) must be an **Even** function]
- $\int_{-a}^{a} f(x)\,dx = 0$ [Here f(x) must be an **odd** function]

- Values to Remember :Sin nπ = 0 & **Cos nπ** = $(-1)^n$

FULL RANGE FOURIER SERIES[Interval of length 2π]

The Fourier series for the function $f(x)$ in the interval **[c , c + 2π]**is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)}$$

Where $\boxed{a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx}$, $\boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx.dx}$ & $\boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx.dx}$

C = 0 **→[0, 2π]**

Remember these formulae as this carries 6M Problem.

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)}$$

Remember this formula as this carries 6M Problem.

Wherea_0= $\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, a_n = $\frac{1}{\pi}\int_0^{2\pi} f(x).Cosnx\,dx$ & b_n = $\frac{1}{\pi}\int_0^{2\pi} f(x).Sin\,nx\,dx$

C = $-\pi \Rightarrow [-\pi, \pi]$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).Cosnx\,dx$ & $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).Sin\,nx\,dx$

__Examples__:

1. Find the Fourier series to represent f(x) = x^2 in the interval $(0, 2\pi)$

Sol. As the given interval is $(0, 2\pi)$, Fourier series becomes -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x).Cosnx\,dx$ & $b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x).Sin\,nx\,dx$

__Step One__ : -

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi}[(2\pi)^3 - 0] = \frac{8}{3}\pi^2$$

$$\Rightarrow \boxed{a_0 = \frac{8}{3}\pi^2}$$

__Step Two__ : -

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\,\underbrace{\cos nx}_{v}\,dx$$

$$= \left[(x^2)\left(\frac{Sin\,nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{Sin\,nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{4}{n^2} \quad \left[\because \begin{matrix}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{matrix}\right] \quad \Rightarrow \boxed{a_n = \frac{4}{n^2}}$$

__Step Three__ : -

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\,\underbrace{\sin nx}_{v}\,d$$

$$= \left[(x^2)\left(-\frac{Cos\,nx}{n}\right) - (2x)\left(-\frac{Sin\,nx}{n^2}\right) + (2)\left(\frac{Cos\,nx}{n^3}\right)\right]_0^{2\pi}$$

$$= -\frac{4\pi}{n} \quad \left[\because \begin{matrix}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{matrix}\right] \quad \Rightarrow \boxed{b_n = -\frac{4\pi}{n}}$$

Finally,
$$\therefore f(x) = x^2 = \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$
$$\Longrightarrow x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

2. Express the function f(x) $= \begin{cases} x & 0 < x < \pi \\ \pi & \pi < x < 2\pi \end{cases}$ as Fourier Series .

Sol. As the given interval is (0, 2π), Fourier series becomes -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Cosnx\,dx$ & $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).Sin\,nx\,dx$

STEP ONE

$$\begin{aligned} a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\mathrm{d}x &= \frac{1}{\pi}\int_0^{\pi} f(x)\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{\pi}{\pi}\left[x\right]_{\pi}^{2\pi} \\ &= \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) + \left(2\pi - \pi\right) \\ &= \frac{\pi}{2} + \pi \\ \text{i.e. } a_0 &= \frac{3\pi}{2}. \end{aligned}$$

STEP TWO

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\cos nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\cos nx\,\mathrm{d}x \\ &= \underbrace{\frac{1}{\pi}\left[\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\mathrm{d}x\right]}_{\text{using integration by parts}} + \frac{\pi}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi} \\ &= \frac{1}{\pi}\left[\frac{1}{n}\left(\pi\sin n\pi - 0\cdot\sin n0\right) - \left[\frac{-\cos nx}{n^2}\right]_0^{\pi}\right] \\ &\qquad + \frac{1}{n}(\sin n2\pi - \sin n\pi) \end{aligned}$$

$$\begin{aligned} \text{i.e. } a_n &= \frac{1}{\pi}\left[\frac{1}{n}\left(0-0\right) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right] + \frac{1}{n}\left(0-0\right) \\ &= \frac{1}{n^2\pi}(\cos n\pi - 1), \quad \text{see TRIG} \\ &= \frac{1}{n^2\pi}((-1)^n - 1), \end{aligned}$$

$$\text{i.e. } a_n = \begin{cases} -\frac{2}{n^2\pi} & , n \text{ odd} \\ 0 & , n \text{ even.} \end{cases}$$

STEP THREE

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} x\sin nx\,\mathrm{d}x + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cdot\sin nx\,\mathrm{d}x \\ &= \underbrace{\frac{1}{\pi}\left[\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi}\left(\frac{-\cos nx}{n}\right)\mathrm{d}x\right]}_{\text{using integration by parts}} + \frac{\pi}{\pi}\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi} \\ &= \frac{1}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}\right] - \frac{1}{n}(\cos 2n\pi - \cos n\pi) \\ &= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \left(\frac{\sin n\pi - \sin 0}{n^2}\right)\right] - \frac{1}{n}(1 - (-1)^n) \\ &= -\frac{1}{n}(-1)^n + \quad 0 \quad - \frac{1}{n}(1 - (-1)^n) \end{aligned}$$

Hence the Fourier series becomes ,

$$f(x) = \frac{1}{2}\left(\frac{3\pi}{2}\right) + \left(-\frac{2}{\pi}\right)\left[\cos x + 0\cdot\cos 2x + \frac{1}{3^2}\cos 3x + \dots\right]$$
$$+\left(-1\right)\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots\right]$$

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3. **Express $f(x) = x-\pi$ as Fourier series in the interval $-\pi < x < \pi$**

Sol Let the function $x-\pi$ be represented by the Fourier series

$$x-\pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

Then

Sep – 1

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} xdx - \pi\int_{-\pi}^{\pi} dx\right]$$
$$= \frac{1}{\pi}\left[0-\pi.2\int_{0}^{\pi} dx\right] \quad (\because x \text{ is odd function})$$
$$= \frac{1}{\pi}\left[-2\pi(x)_0^{\pi}\right]$$
$$= -2(\pi-0) = -2\pi \text{ and}$$

Step - 2

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\cos nx.dx$$
$$= \frac{1}{x}\left[\int_{-\pi}^{\pi} x\cos nx.dx - \pi\int_{-\pi}^{\pi}\cos nx.dx\right]$$
$$= \frac{1}{\pi}\left[0-2\pi\int_0^{\pi}\cos nx.dx\right]$$

($\because x\cos nx$ is odd function and $\cos nx$ is even function)

$$\therefore a_n = -2\int_0^{\pi}\cos nx.dx$$
$$= -2\left(\frac{\sin nx}{n}\right)_0^{\pi}$$
$$= \frac{-2}{n}(\sin n\pi - \sin 0)$$
$$= \frac{-2}{n}(0-0) = 0 \quad for \ n = 1,2,3\dots\dots$$

Sep – 3

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\sin nx.dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx - \pi\int_{-\pi}^{\pi}\sin nx.dx\right]$$
$$= \frac{1}{\pi}\left[2\int_0^{\pi} x\sin nx.dx - \pi(0)\right]$$
$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}+0\right)-(0+0)\right]$$
$$= \frac{-2}{\pi}\cos n\pi = \frac{-2}{n}(-1)^n$$
$$= \frac{2}{n}(-1)^{n+1} \ \forall n = 1,2,3\dots\dots$$

Substituting the values of a_0, a_n, b_n in (1),

We get ,

$$\boxed{\begin{aligned} x-\pi &= -\pi+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{\pi}\sin nx \\ &= -\pi+2\left[\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+......\right] \end{aligned}}$$

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4. Find the Fourier Series of the periodic function defined as $f(x)=\begin{cases} -\pi & -\pi<x<0 \\ x & 0<x<\pi \end{cases}$

Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+------=\frac{\pi^2}{8}$

Sol. Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \rightarrow (1)$ Then

Step 1 :

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)\,dx+\int_{0}^{\pi}x\,dx\right] \\ &= \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0}+\left(\frac{x^2}{2}\right)_{0}^{\pi}\right] \\ &= \frac{1}{\pi}\left[-\pi^2+\frac{\pi^2}{2}\right]=\frac{1}{\pi}\left[\frac{-\pi^2}{2}\right]=\frac{-\pi}{2} \end{aligned}$$

Step 2 :

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)\cos nx.dx+\int_{0}^{\pi}x\cos nx.dx\right] \\ &= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0}+\left(x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right)_{0}^{\pi}\right] \\ &= \frac{1}{\pi}\left[0+\frac{1}{n^2}\cos n\pi-\frac{1}{\pi n^2}\right] \end{aligned}$$

$$=\frac{1}{\pi}\left[\frac{(-1)^n}{n^2}+\frac{1}{n^2}\right]=\frac{1}{\pi n^2}[(-1)^n-1]$$

$$\Rightarrow \boxed{a_n=\frac{1}{\pi n^2}[(-1)^n-1]}$$

Step 3 :

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx.dx \\ &= \frac{1}{\pi}\left[\int_{\pi}^{0}(-\pi)\sin nx.dx+\int_{0}^{\pi}x\sin nx.dx\right] \\ &= \frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0}+\left(-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right)_{0}^{\pi}\right] \\ &= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi)-\frac{\pi}{n}\cos n\pi\right] \\ &= \frac{1}{n}(1-2\cos n\pi) \end{aligned}$$

$$b_1=3, b_2=\frac{-1}{2}, b_3=1, b_4=\frac{-1}{4} ....$$

Substituting the values of a_0, a_n *and* b_n in (1), we get

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + --- \right) + \left(3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + -- \right)$$

Deduction: Put $x = 0$ in the above function $f(x)$, we get

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

Since, $f(x)$ is discontinuous at $x = 0$, $\begin{matrix} f(0-0) = -\pi \\ f(0+0) = 0 \end{matrix}$

$$\Rightarrow f(0) = \frac{1}{2}[f(0-0) + f(0+0)]$$

$$\Rightarrow f(0) = \frac{1}{2}(-\pi) = -\frac{\pi}{2}$$

Hence, $f(0) = -\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$ $\quad \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Even and Odd Functions:-

A function $f(x)$ is said to be even *if* $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$

Example:- $x^2, x^4 + x^2 + 1, e^x + e^{-x}$ are even functions& $x^3, x, \sin x, \operatorname{cosec} x$ are odd functions.

Note 1 :-

1. Product of two even (or) two odd functions will be an even function
2. Product of an even function and an odd function will be an odd function

Note 2:- $\int_{-a}^{a} f(x)dx = 0$ when $f(x)$ is an odd function

$= 2\int_0^a f(x)dx$ When $f(x)$ is even function

Fourier series for even and odd functions

We know that a function $f(x)$ defined in $(-\pi, \pi)$ can be represented by the Fourier series

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx}$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$

And $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$

Case 1 :- when $f(x)$ is **even function**

Since $\cos nx$ is an even function,

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx}$$

➔ $f(x)\cos nx$ is also an even function.

Hence $$\boxed{\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx.dx \end{aligned}}$$

Since $\sin nx$ is an odd function,

➔ $f(x)\sin nx$ is an odd function

$$\boxed{\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = 0}.$$

Case 2:- when $f(x)$ is an **odd function**

since $f(x)$ is an odd function

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0}$$

➔Since $\cos nx$ is an even function, $f(x)\cos nx$ is an odd function and

Hence $$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx = 0}$$

➔Since $\sin nx$ is odd function; $f(x)\sin nx$ is an even function

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = \frac{2}{\pi}\int f(x)\sin nx.dx$$

Examples:-

1. Expand the function $f(x) = x^2$ as a Fourier series in $(-\pi,\pi)$, hence deduce that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ---- = \frac{\pi^2}{12}$$

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$ ➔ ***f(x) is an even function.***

Hence in its Fourier series expansion, the sine terms are absent

$$\therefore x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$\therefore\ b_n = 0$

No need to find

Step 1 :

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} x^2 dx$$

$$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3}$$

Step 2 :

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx.dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx.dx$$
$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$
$$= \frac{2}{\pi}\left[0 + 2\pi\frac{\cos nx}{n^2} + 2.0\right]$$

$$= \frac{4\cos n\pi}{n^2} = \frac{4}{n^2}(-1)^n$$

Substituting the values of a_0 *and* a_n , we get

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$$
$$= \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$
$$= \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + ---\right)$$

Deductions:- Putting $x = 0$ in (4), we get

$$0 = \frac{\pi^2}{3} - 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ---\right)$$
$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ---- = \frac{\pi^2}{12}$$

FULL RANGE FOURIER SERIES[Interval of length 2l]

The Fourier series for the function $f(x)$ in the interval **[c, c + 2*l*]**is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{l} + b_n Sin\frac{n\pi x}{l}\right)}$$

Where $\boxed{a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx}$, $\boxed{a_n = \frac{1}{l}\int_c^{c+2l} f(x)Cos\frac{n\pi x}{l}dx}$ $\boxed{b_n = \frac{1}{l}\int_c^{c+2l} f(x)\,Sin\frac{n\pi x}{l}dx}$

Remember this formula as this carries 6M Problem.

C = 0 ➔**[0, 2*l*]**

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{l} + b_n Sin\frac{n\pi x}{l}\right)}$$

Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$, $a_n = \frac{1}{l}\int_0^{2l} f(x)\,Cos\frac{n\pi x}{l}\,dx$ & $b_n = \frac{1}{l}\int_0^{2l} f(x)\,Sin\frac{n\pi x}{l}\,dx$

C = −*l*➔**[−*l*, *l*]**

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n Cos\frac{n\pi x}{l} + b_n Sin\frac{n\pi x}{l}\right)}$$

Where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$, $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\,Cos\frac{n\pi x}{l}\,dx$ & $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\,Sin\frac{n\pi x}{l}\,dx$

Examples:-

1. Express $f(x) = x^2$ as a Fourier series in $[-l, l]$

SEE FOR ***EVEN*** OR ***ODD*** FUNCTION AS THE INTERVAL IS FROM – VALUE TO + VALUE

Sol $f(-x) = f(-x)^2 = x^2 = f(x)$

Therefore $f(x)$ is an even function

Hence the Fourier series of $f(x)$ in $[-l, l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$where\ a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$\boxed{hence\ a_0 = \frac{2}{l}\int_0^l x^2 dx = \frac{2}{l}\left(\frac{x^3}{3}\right)_0^l = \frac{2l}{3}}$$

$$= \frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}dx \qquad also\ a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[2x\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^l \qquad = \frac{2}{l}\left[x^2\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right] - 2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

Since the first and last terms vanish at both upper and lower limits

$$\therefore a_n = \frac{2}{l}\left[2l\frac{\cos n\pi}{n^2\pi^2/l^2}\right] = \frac{4l^2\cos n\pi}{n^2\pi^2}$$

$$= \frac{(-1)^n 4l^2}{n^2\pi^2}$$

Substituting these values in (1), we get

$$x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n 4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2} - \frac{\cos(2\pi x/l)}{2^2} + \frac{\cos(3\pi x/l)}{3^2} - - - - - -\right]$$

---

2. Find a Fourier series with period 3 to represent $f(x) = x + x^2 \;\; in\ (0,3)$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \to (1)$

Here $2l = 3$, $l = 3/2$. Hence (1) becomes

$$f(x) = x + x^2$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3} + b_n\sin\frac{2n\pi x}{3}\right) \to (2)$$

Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$

$$= \frac{2}{3}\int_0^3 (x+x^2)\,dx = \frac{2}{3}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^3 = 9$$

and $a_n = \frac{1}{l}\int_0^2 f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{3}\int_0^3 (x+x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

Using bracketing method, we obtain

$$a_n = \frac{2}{3}\left[\frac{3}{4n^2\pi^2 - 4n^2\pi^2}\right] = \frac{2}{3}\left(\frac{54}{9n^2\pi^2}\right) = \frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx \qquad = \frac{2}{3}\int_0^3 (x+x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

Substituting the values of a's and b's in (2) we get

$$x+x^2=\frac{9}{2}+\frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right)-\frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

Half –Range Fourier Series(Interval of length *l*) ➔[0,*l*]

Remember these formulae as this carries 6M Problem.

Part – B [3Q – (b)]

The Cosine series

$$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.Cos\frac{n\pi x}{l}}$$

Where

$$a_0=\frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n=\frac{2}{l}\int_0^l f(x).Cos\frac{n\pi x}{l}$$

The sine series

$$\boxed{f(x)=\sum_{n=1}^{\infty}b_n.Sin\frac{n\pi x}{l}}$$

Where

$$b_n=\frac{2}{l}\int_0^l f(x).Sin\frac{n\pi x}{l}$$

Note:-

1) Suppose $f(x)=x \text{ in } [0,\pi]$, it can have Fourier cosine series expansion as well as Fourier sine series expansion in $[0,\pi]$

2) If $f(x)=x^2 \text{ in } [0,\pi]$, can have Fourier cosine series as well as sine series

Examples:-

1. Find the **half range sine** series for $f(x)=x(\pi-x) \text{ in } 0<x<\pi$. Deduce that

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+-----=\frac{\pi^3}{32}$$

Half range ➔

$(0,l)$ means $(0,\pi)$

$\therefore l=\pi$

Ans. The ***Fourier sine series*** expansion of $f(x)$ in $(0,\pi)$ is

$$f(x)=x(\pi-x)=\sum_{n=1}^{\infty}b_n\sin nx$$

$$where\ b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx.dx$$

$$hence\ b_n=\frac{2}{\pi}\int_0^{\pi}x(\pi-x)\sin nx.dx=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\sin nx.dx$$

$$=\frac{2}{\pi}\left[(\pi x-x^2)\left(\frac{-\cos nx}{n}\right)-(\pi-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{2}{n^3}(1-\cos n\pi)\right]$$

$$=\frac{4}{n\pi^3}\left(1-(-1)^n\right)$$

Hence
$$x(\pi-x)=\sum_{n=1,3,5...}\frac{8}{\pi n^3}\sin nx\ (or)$$
$$x(\pi-x)=\frac{8}{\pi}\left(\sin x+\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}+----\right)\rightarrow(1)$$

Deduction:- Putting $x=\frac{\pi}{2}$ in (1), we get

$$\frac{\pi}{2}\left(x-\frac{\pi}{2}\right)=\frac{8}{\pi}\left(\sin\frac{\pi}{2}+\frac{1}{3^3}\sin\frac{3\pi}{2}+\frac{1}{5^3}sin\frac{5\pi}{2}+--\right)$$

$$\frac{\pi^2}{4}=\frac{8}{\pi}\left[1+\frac{1}{3^3}\sin\left(\pi+\frac{\pi}{2}\right)+\frac{1}{5^3}\sin\left(2\pi+\frac{\pi}{2}\right)+\frac{1}{7^3}\sin\left(3\pi+\frac{\pi}{2}\right)+---\right]$$

Hence

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+-----=\frac{\pi^3}{32}$$

---

2. Find the **half- range sine** series of $f(x)=1\ in\ [0,l]$

Ans. The Fourier sine series of $f(x)\ in\ [0,l]$ is given by $f(x)=1=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$

$$here\ b_n=\frac{2}{l}\int_0^{l}f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^{l}1.\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)_0^l$$

$$=\frac{2}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]_0^l$$

$$=\frac{2}{n\pi}\left(-\cos n\pi+1\right)$$

$$=\frac{2}{n\pi}\left[(-1)^{n+1}+1\right]$$

$\therefore b_n=0$ when n is even

$=\frac{4}{n\pi}$, when n is odd

Hence the required Fourier series is $f(x)=\sum_{n=1,3,5---}^{\infty}\frac{4}{n\pi}\sin\frac{n\pi x}{l}$

3. Find the **half – range cosine** series expansion of $f(x)=\sin\left(\frac{\pi x}{l}\right)$ in the range $0<x<l$

Sol.Half Range Cosine series in $(0,l)$ is given by $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n.Cos\frac{n\pi x}{l}$

$$where\ a_0=\frac{2}{l}\int_0^l f(x)dx=\frac{2}{l}\int_0^l\sin\frac{\pi x}{l}dx$$

$$=\frac{2}{l}\left[\frac{-\cos\pi x/l}{\pi/l}\right]_0^l$$

$$=\frac{2}{l}(\cos\pi-1)=\frac{4}{\pi}\ and$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{1}{l}\int_0^l\left[\frac{\sin(n+1)\pi x}{l}-\frac{\sin(n-1)\pi x}{l}\right]dx$$

$$=\frac{1}{l}\left[-\frac{-\frac{\cos(n+1)\pi x}{l}}{(n+1)\pi/l}+\frac{\cos(n-1)\pi x/l}{(n-1)\pi/l}\right]_0^l$$

$$=\frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

When n is odd

$$a_n=\frac{1}{\pi}\left[\frac{-1}{n+1}+\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]=0$$

When n is even

$$a_n=\frac{1}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

$$=\frac{-4}{\pi(n+1)(n-1)}$$

$$\therefore \sin\left(\frac{\pi x}{l}\right)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos(2\pi x/l)}{1.3}+\frac{\cos(4\pi x/l)}{3.5}+----\right]$$

=======*========**=======*========**=======*========**=======*========

SECTION-A

Objective Questions

1. If $f(x)=\begin{cases}1+\dfrac{2x}{\pi} & -\pi\le x\le 0\\ 1-\dfrac{2x}{\pi} & 0\le x\le\pi\end{cases}$ Then $f(x)$ is _________ function [ ]

   a) Odd b) even c) periodic d) none

2. If the Fourier series for the function $f(x)$ defined in $[-\pi,\pi]$ then a_n = ________
3. The Fourier constant b_n for $f(x)=x\sin x$ in $[-\pi,\pi]$ is______
4. If $f(x) = x^2$ in $(-l, l)$then a_0& $b_1$are _______________
5. If $f(x) = |x|$in $(-\pi, \pi)$ then a_1& b_1 are ______________

6. In Fourier expansion of $f(x)=x+x^2$ in $(-\pi,\pi)$ the value of a_n is []

   a) $\frac{2}{n^2}(-1)^4$ b) $\frac{4}{n^2}(-1)^n$ c) 0 d) none

7. If $f(x)=x\cos x$ in $(-\pi,\pi)$ then a_n is []

   a) 1 b) 2 c) 3 d) 0

8. If $f(x)$ is expanded as a Fourier series in $(0,2\pi)$ then a_0 =____________ [ ]

   a) $\frac{1}{\pi}\int_0^{2\pi} f(x)dx$ b) $\frac{1}{\pi}\int_0^{\pi} f(x)dx$ c) $\frac{2}{\pi}\int_0^{2\pi} f(x)dx$ d) none

9. Fourier sine series for $f(x) = x$ in $(0,\pi)$ is ____________________

10. If $f(x) =$ $sinx$ in $-\pi < x < \pi$ then a_0=_________________

11. In Fourier series expansion of $f(x) = coshx$ in $(-4,4)$ the Fourier co efficient a_1 is ____________________

12. If $f(x)$ is expanded as a Fourier series in $[0,2\pi]$ then b_n =__________ [ ]

a) $\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx.dx$ b) $\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx.dx$

c) $\frac{2}{\pi}\int_0^{2\pi} f(x)\sin nx.dx$ d) none

13. 10. If $f(x)=1+\sin x$ in $(-1,1)$ is expressed as a Fourier series then the Value of b_n =______[ ]

a) 0 b) 1 c) 2 d) none

SECTION-B

II) Level Two Questions:

1. Obtain Fourier Series for the function $f(x) = \begin{cases} x, & if\ 0 < x < \pi \\ 2\pi - x, & if\ \pi < x < 2\pi \end{cases}$

And hence deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

2. Obtain the Fourier series to represent $x - x^2$ in $(-\pi,\pi)$ and deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$
3. If $f(x) = x^2$, $-l < x < l$. Obtian Fourier Series and deduce that $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$
4. Expand $f(x) = e^{-x}$ as a Fourier seriesin$(-1,1)$.
5. Obtain Fourier series to represent the function $f(x) = |x|$ in $(-\pi,\pi)$and deduce that

$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

6. Obtain the Fourier series expansion of f(x) given that $f(x) = (\pi - x)^2 in\ 0 < x < 2\pi$ and deduce that $1/1^2 + 1/2^2 + 1/3^2 + \ldots\ldots\ldots\ldots = \pi^2/6$
7. Find a Fourier series to represent the function $f(x) = e^x$ for $-\pi < x < \pi$ and hence derive a series for $\pi/sinh\pi$
8. Find the Fourier series of the periodic function $f(x) = \begin{cases} -\pi\ , -\pi < x < \pi \\ x\ ,\ 0 < x < \pi \end{cases}$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\ldots\ldots\ldots. = \frac{\pi^2}{8}$

9. Find the half-range cosine series and sine series for $f(x) = x$ in $0 < x < \pi$ hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\ldots\ldots\ldots = \frac{\pi^2}{8}$
10. Find the Fourier series expansion for $f(x) = \begin{cases} 2, -2 < x < 0 \\ x, 0 < x < 2 \end{cases}$
11. Find the Fourier series expansion for the function $f(x) = x - x^2 in\ (-1,1)$

12. Show that the Fourier series expansion of f(x) = 1 in 0<x<1 and f(x) = 2 in 1<x<3 with f(x+3) = f(x) is $\frac{5}{3}+\frac{9}{4\pi}\left[\frac{\sqrt{3}}{2}\cos\left(\frac{3\pi x}{2}\right)-\frac{\sqrt{3}}{4}\cos 3\pi x+....\right]+\frac{9}{4\pi}\left[-\frac{3}{2}\sin\left(\frac{3\pi x}{2}\right)-\frac{3}{4}\sin 3\pi x+....\right]$ (DEC 2015)

13. Find the half-range cosine series for the function f(x)= $\begin{cases} kx, 0 \le x \le \frac{l}{2} \\ k(l-x), \frac{l}{2} \le x \le l \end{cases}$

14. Express f(x) = x as a half range sine series in 0 < x < 2.

15. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$

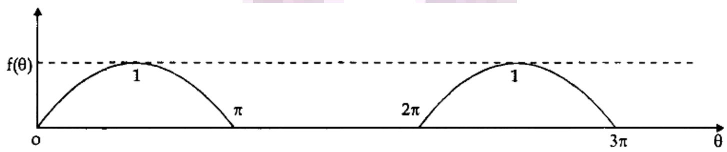
Hence show that $\sum_{n=1}^{\infty}\frac{1}{(2x-1)^2} = \frac{\pi^2}{8}$

SECTION-C

C. Questions testing the analyzing / evaluating ability of students

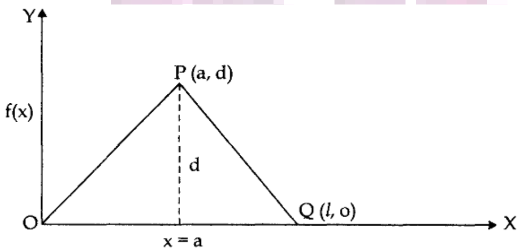
Level Three Questions:

1. An alternating current after passing through a rectifier has form $i = \begin{cases} l.\sin\theta & 0<\theta<\pi \\ 0 & \pi<\theta<2\pi \end{cases}$.



Find the Fourier series of the function.

2. Find the half period series for f(x) given in the range (0,L) by the graph OPQ as shown in the following fig.



Hint. $f(x) = \begin{cases} \frac{xd}{a}, & 0<x<a \\ \frac{d(l-x)}{l-a}, & a<x<l \end{cases}$.

Gate Previous year Questions :

2016 Let $f(x)$ be a real, periodic function satisfying $f(-x) = -f(x)$. The general form of its Fourier series representation would be

(A) $f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx)$

(B) $f(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$

(C) $f(x) = a_0 + \sum_{k=1}^{\infty} a_{2k} \cos(kx)$

(D) $f(x) = \sum_{k=0}^{\infty} a_{2k+1} \sin(2k+1)x$

2015

The signum function is given by

$$sgn(x) = \begin{cases} \frac{x}{|x|}; x \neq 0 \\ 0; x = 0 \end{cases}$$

The Fourier series expansion of $sgn(\cos(t))$ has

(A) only sine terms with all harmonics.
(B) only cosine terms with all harmonics.
(C) only sine terms with even numbered harmonics.
(D) only cosine terms with odd numbered harmonics.

Options :

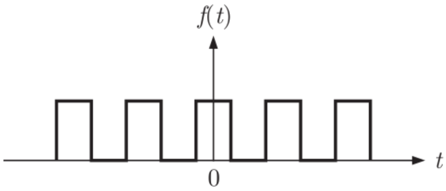
1. ✖ A
2. ✖ B
3. ✖ C
4. ✔ D

2012 Let $x(t)$ be a periodic signal with time period T, Let $y(t) = x(t - t_0) + x(t + t_0)$ for some t_0. The Fourier Series coefficients of $y(t)$ are denoted by b_k. If $b_k = 0$ for all odd k, then t_0 can be equal to

(A) $T/8$ (B) $T/4$

(C) $T/2$ (D) $2T$

2011 The Fourier series expansion $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t$ of the periodic signal shown below will contain the following nonzero terms



(A) a_0 and $b_n, n = 1,3,5,\ldots\infty$ (B) a_0 and $a_n, n = 1,2,3,\ldots\infty$

(C) $a_0\, a_n$ and $b_n, n = 1,2,3,\ldots\infty$ (D) a_0 and $a_n\; n = 1,3,5,\ldots\infty$

The period of the signal $x(t) = 8\sin\left(0.8\pi t + \frac{\pi}{4}\right)$ is

(A) 0.4π s (B) 0.8π s

(C) 1.25 s (D) 2.5 s

2010

2009

The Fourier Series coefficients of a periodic signal $x(t)$ expressed as $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$ are given by $a_{-2} = 2 - j1$, $a_{-1} = 0.5 + j0.2$, $a_0 = j2$, $a_1 = 0.5 - j0.2$, $a_2 = 2 + j1$ and $a_k = 0$ for $|k| > 2$

Which of the following is true ?

(A) $x(t)$ has finite energy because only finitely many coefficients are non-zero

(B) $x(t)$ has zero average value because it is periodic

(C) The imaginary part of $x(t)$ is constant

(D) The real part of $x(t)$ is even

2008

Let $x(t)$ be a periodic signal with time period T, Let $y(t) = x(t - t_0) + x(t + t_0)$ for some t_0. The Fourier Series coefficients of $y(t)$ are denoted by b_k. If $b_k = 0$ for all odd k, then t_0 can be equal to

(A) $T/8$ (B) $T/4$

(C) $T/2$ (D) $2T$

2007

A signal $x(t)$ is given by

$$x(t) = \begin{cases} 1, -T/4 < t \leq 3T/4 \\ -1, 3T/4 < t \leq 7T/4 \\ -x(t+T) \end{cases}$$

Which among the following gives the fundamental fourier term of $x(t)$?

(A) $\frac{4}{\pi}\cos\left(\frac{\pi t}{T} - \frac{\pi}{4}\right)$ (B) $\frac{\pi}{4}\cos\left(\frac{\pi t}{2T} + \frac{\pi}{4}\right)$

(C) $\frac{4}{\pi}\sin\left(\frac{\pi t}{T} - \frac{\pi}{4}\right)$ (D) $\frac{\pi}{4}\sin\left(\frac{\pi t}{2T} + \frac{\pi}{4}\right)$

****

UNIT IV

CALCULUS

Rolle's Mean Value Theorem:

Suppose $f(x)$ is a function that satisfies all of the following.

1. $f(x)$ is continuous on the closed interval [*a*,*b*].
2. $f(x)$ is differentiable on the open interval (*a*,*b*).
3. $f(a) = f(b)$

Then there is a number *c* such that $a < c < b$ and $f'(c) = 0$. Or, in other words $f(x)$ has a critical point in (*a*,*b*).

Let's take a look at a quick example that uses Rolle's Theorem.

Example 1 Show that $f(x) = 4x^5 + x^3 + 7x - 2$ has exactly one real root.

Solution

From basic Algebra principles we know that since $f(x)$ is a 5th degree polynomial it will have five roots. What we're being asked to prove here is that only one of those 5 is a real number and the other 4 must be complex roots.

First, we should show that it does have at least one real root. To do this note that $f(0) = -2$ and that $f(1) = 10$ and so we can see that $f(0) < 0 < f(1)$. Now, because $f(x)$ is a polynomial we know that it is continuous everywhere and so a number *c* such that $0 < c < 1$ and $f(c) = 0$. In other words $f(x)$ has at least one real root.

We now need to show that this is in fact the only real root. To do this we'll use an argument that is called contradiction proof. What we'll do is assume that $f(x)$ has at least two real roots. This means that we can find real numbers *a* and *b* (there might be more, but all we need for this particular argument is two) such that $f(a)=f(b)=0$. But if we do this then we know from Rolle’s Theorem that there must then be another number *c* such that $f'(c)=0$ $f'(c)=0$. This is a problem however. The derivative of this function is,

$$f'(x)=20x^4+3x^2+7$$

Because the exponents on the first two terms are even we know that the first two terms will always be greater than or equal to zero and we are then going to add a positive number onto that and so we can see that the smallest the derivative will ever be is 7 and this contradicts the statement above that says we MUST have a number *c* such that $f'(c)=0$. We reached these contradictory statements by assuming that $f(x)$ has at least two roots. Since this assumption leads to a contradiction the assumption must be false and so we can only have a single real root.

Geometrical Interpretation of Rolle’s Mean Value Theorem:

The proof of the mean value theorem is very simple and intuitive. We just need our intuition and a little of algebra. To prove it, we'll use a new theorem of its own: **Rolle's Theorem.**

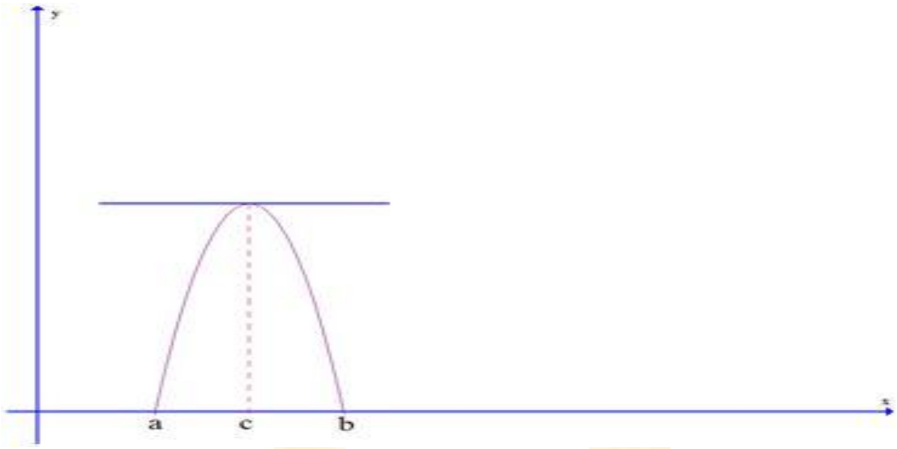
This theorem says that given a continuous function g on an interval [a,b], such that g(a)=g(b), then there is some c, such that:

$$a < c < b$$

And:

$$f'(c) = 0$$

Graphically, this theorem says the following:



Given a function that looks like that, there is a point c, such that the derivative is zero at that point. That implies that the tangent line at that point is horizontal. Why? Because the derivative is the slope of the tangent line. Slope zero implies horizontal line.

Lagrange's Mean Value Theorem

Suppose $f(x)$ is a function that satisfies both of the following.

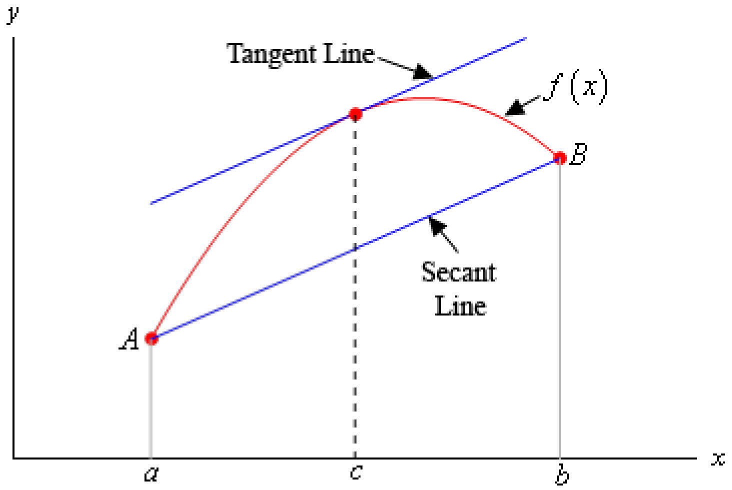
1. $f(x)$ is continuous on the closed interval [*a*,*b*].
2. $f(x)$ is differentiable on the open interval (*a*,*b*).

Then there is a number *c* such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or,

$$f(b) - f(a) = f'(c)(b - a)$$



Let’s now take a look at a couple of examples using the Mean Value Theorem.

Example: Determine all the numbers *c* which satisfy the conclusions of the Mean Value Theorem for the following function.

$$f(x) = x^3 + 2x^2 - x \quad \text{on} \quad [-1,2]$$

Solution

There isn’t really a whole lot to this problem other than to notice that since $f(x)$ is a polynomial it is both continuous and differentiable (*i.e.* the derivative exists) on the interval given.First let’s find the derivative.

$$f'(x) = 3x^2 + 4x - 1$$

Now, to find the numbers that satisfy the conclusions of the Mean Value Theorem all we need to do is plug this into the formula given by the Mean Value Theorem.

$$f'(c)=\frac{f(2)-f(-1)}{2-(-1)}$$

$$3c^2+4c-1=\frac{14-2}{3}=\frac{12}{3}=4$$

Now, this is just a quadratic equation,

$$3c^2+4c-1=4$$
$$3c^2+4c-5=0$$

Using the quadratic formula on this we get,

$$c=\frac{-4\pm\sqrt{16-4(3)(-5)}}{6}=\frac{-4\pm\sqrt{76}}{6}$$

So, solving gives two values of *c*.

$$c=\frac{-4+\sqrt{76}}{6}=0.7863 \qquad c=\frac{-4-\sqrt{76}}{6}=-2.1196$$

Notice that only one of these is actually in the interval given in the problem. That means that we will exclude the second one (since it isn't in the interval). The number that we're after in this problem is,

$$c=0.7863$$

Be careful to not assume that only one of the numbers will work. It is possible for both of them to work.

Cauchy's Mean Value Theorem:-

Statement:- **If two functions f(x) and g(x) are derivable in a closed interval [a,b] and g′(x) ≠ 0 for any value of x in [a,b] then there exists at least one value 'c' of x belonging to the open interval (a,b) such that**

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Fact

1. If $f'(x)>0$ for every *x* on some interval *I*, then $f(x)$ is increasing on the interval.
2. If $f'(x)<0$ for every *x* on some interval *I*, then $f(x)$ is decreasing on the interval.
3. If $f'(x)=0$ for every *x* on some interval *I*, then $f(x)$ is constant on the interval.

First Derivative Test

Suppose that $x=c$ is a critical point of $f(x)$ then,

1. If $f'(x)>0$ to the left of $x=c$ and $f'(x)<0$ to the right of $x=c$ then $x=c$ is a relative maximum.
2. If $f'(x)<0$ to the left of $x=c$ and $f'(x)>0$ to the right of $x=c$ then $x=c$ is a relative minimum.
3. If $f'(x)$ is the same sign on both sides of $x=c$ then $x=c$ is neither a relative maximum nor a relative minimum.

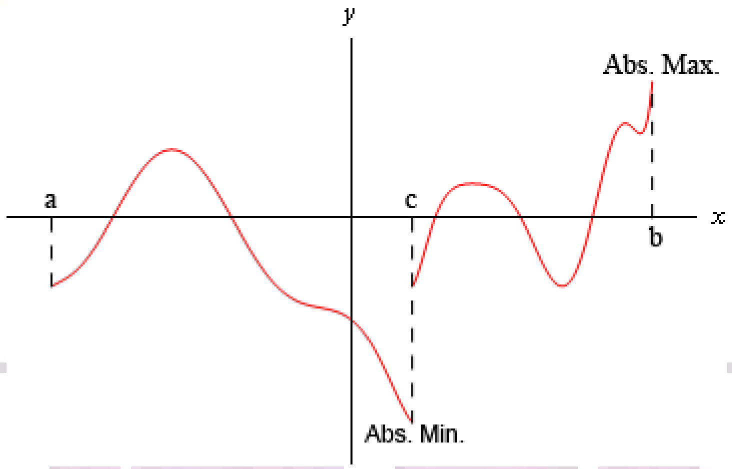
Definition

1. We say that *f(x)* has an **absolute (or global) maximum** at if for every *x* in the domain we are working on.

2. We say that *f(x)* has a **relative (or local) maximum** at if for every *x* in some open interval around .

3. We say that *f(x)* has an **absolute (or global) minimum** at if for every *x* in the domain we are working on.

4. We say that *f(x)* has a **relative (or local) minimum** at if for every *x* in some open interval around .



Example: **Verify Cauchy's mean value theorem for** $f(x)=\sqrt{x}$ **and** $g(x)=\frac{1}{\sqrt{x}}$ **in [a,b] when 0<a<b.**

Sol: -

Given $f(x)=\sqrt{x}.g(x)=\frac{1}{\sqrt{x}}$

$\because f(x)-g(x)$ are conts in [a,b] & f(x), g(x) are derivable in (a,b)

$\therefore f^1(x)=\dfrac{1}{2\sqrt{x}}$ & $g^1(x)=\dfrac{1}{2x\sqrt{x}}$ exists in (a,b)

Also $g^1(x)\neq 0 \ \forall\ x\in(a,b)\subseteq R^+$

$\therefore f(x)g(x)$ are satisfics all conditions of cauchy's mean value Theorem.

VERIFICATION: -

By Cauchy's mean value Theorem. Is al least one $c\in(a,b)$ such that $\dfrac{f^1(c)}{g(c)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$

$$\therefore \frac{\frac{1}{2}\sqrt{c}}{-\frac{1}{2c\sqrt{c}}}=\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}\Rightarrow -c=\frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}}=-\sqrt{ab}\Rightarrow \therefore c=\sqrt{b}\mathrm{E}(a,b)$$

Here c is Geometric mean of a & b

$\therefore$ Cauchy's mean value theorem verified

Generalised Mean Value Theorems

Taylor's theorem

If a function f(x) is such that f(x), $f^1(x)$,……. $f^{(n-1)}(x)$ are continuous in [a,a+h] and $f^{(n)}(x)$ exists for all $x\in(a,a+h)$ then there exists $\theta\in(0,1)$ such that

$$f(a+h)=f(a)+hf^1(a)+\frac{h^2}{\angle 2}f''(a)+----\frac{h^{n-1}}{\angle(n-1)}f^{(n-1)}(a)+R_n$$

Here $R_n=\dfrac{h^n(1-\theta)^{n-p}}{(n-1)!\,p}f^{(n)}(a+\theta h)$ is called the Taylor's reminder after n terms

If p = 1 we get **Cauchy's form of Reminder** and

If p = n we get **Lagrange's form of reminder**

Maclaurin's theorem:

If a function f(x) is such that

(i) f(x), $f^1(x)$,……. $f^{(n-1)}(x)$ are continuous in [0,x] and (ii) $f^{(n)}(x)$ exists for all

$x\in(0,x)$ then there exists $\theta\in(0,1)$ such that

$$f(x)=f(0)+xf^1(0)+\frac{x^2}{\angle 2}f''(0)+-----+\frac{x^{n-1}}{\angle(n-1)}f^{(n-1)}(0)+R_n$$

where $R_n=\dfrac{x^n(1-\theta)^{n-p}}{(n-1)!\,p}f^{(n)}(\theta x)$ is called the Maclaurin's reminD.E.r after n terms

If p = 1 we get **Cauchy's form** and

If p = n we get **Lagrange's form of reminder**.

Verify Taylor's Theorem for $f(x)=(1-x)^{5/2}$ with Lagrange's form of remainder up to 2 terms in the interval [0,1]

Sol: -

Given $f(x)=(1-x)^{5/2} \Rightarrow f^1(x)=\frac{-5}{2}(1-x)^{3/2}$

$\because f^1(x)$ is polynomias in x

$\because f^1(x)$ is conts in [0,1] & $f^1(x)$ is derivable in (0,1)

They f(x) satisfies all condition of Taylor's theorem .

By Taylor's theorem with Lagrange's form of remainder is at least one $c\in(0,1)$ such that

$$f(b)=f(a)+\frac{(b-a)}{1!}f^1(a)+R_n$$

When $R_n=\dfrac{(b-a)^n f^n(c)}{n!}$

Here n = p = 2; a = 0, b = 1

$$f(1)=f(0)+f^1(0)+\frac{f^{11}(c)}{2!}\Rightarrow f(0)=1;\ f(1)=0$$

Now $f^{1}(x)=\frac{-5}{2}(1-x)^{3/2}$, $f^{1}(0)=\frac{-5}{2}\Rightarrow f^{11}(x)=\frac{15}{4}(1-x)^{\frac{1}{2}}\ f^{11}(c)=\frac{15}{4}(1-c)^{\frac{1}{2}}$

$\therefore 0=1-\frac{5}{2}+\frac{15}{8}(1-c)^{1/2}\Rightarrow(1-c)^{1/2}=\frac{3}{2}.\frac{8}{15}=\frac{4}{5}\Rightarrow 1-c=\frac{16}{25}\Rightarrow c=1-\frac{16}{25}=\frac{9}{25}=0.36$

Hence c = 0.36 $\in(0,1)$

$\therefore$ Taylor's Theorem is verified.

Show that $Cosx=1-\frac{x^2}{2!}+\frac{x^4}{4!}---$

Sol : -wkt From Maclaurin's series

$$f(x)=f(0)+x.f^{1}(0)+\frac{x^2}{2!}f^{11}(0)+\frac{x^3}{3!}f^{111}(0)+\frac{x^4}{4!}f^{4}(0)+--$$

Given

$f(x)=\cos x$ $\qquad f(0)=1$

$f^{1}(x)=-\sin x$ $\qquad f^{1}(0)=0$

$f^{11}(x)=-Cosx$ $\qquad f^{11}(0)=-1$

$f^{111}(x)=\sin x$ $\qquad f^{111}(0)=0$

$f^{4}(x)=Cosx$ $\qquad f^{4}(0)=1$

$------$

Now $Cosx=1-\frac{x^2}{2!}+\frac{x^4}{4!}+--$

Beta – Gamma Functions

- **Gamma Function:** The definite integral $\int_0^\infty e^{-x}x^{n-1}dx$ n > 0 is a function of n. It is called the Gamma function or Euler integral of the second kind and is denoted by $\Gamma(n)$.

- $\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$
- Reduction formula for $\Gamma(n)$:

  $\Gamma(n+1) = n\Gamma(n)$

  If n is a positive integer then $\Gamma(n+1)$ =n(n-1)(n-2)……3.2.1. $\Gamma(1)$ = n**!**

  If n is a positive fraction then $\Gamma(n)$ = (n-1) $\Gamma(n-1)$

  = (n-1)(n-2) $\Gamma(n-2)$ ……

  This process is to be continued as long as the factors remain positive and each factor is obtained by subtracting from the preceding factor and is to be multiplied by of the last factor.

  If n is a negative fraction then $\Gamma(n) = \dfrac{\Gamma(n+k)}{n(n+1)....(n+k)}$ where k is the least positive integer such that n+k+1>0

- $\Gamma(0)$ is not defined.
- $\Gamma(n)$ is not defined when n is negative integer.
- **Beta Function:** The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ m>0,n>0 which is a function of m and n is called beta function or the Eulearian integral of first kind and is denoted by $\beta(m,n)$

$\beta(m,n)$ = $\beta(n,m)$ - **Symmetry**

Beta in terms of Trigonometric functions:

$$\beta(m,n) = \mathbf{2}\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$$

Relation Between Beta and Gamma functions:

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\int_0^{\pi/2} \sin^p x \cos^q x dx = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma(q+1)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

$\Gamma(1/2) = \sqrt{\pi}$ =1.772

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\frac{\sqrt{\pi}}{2}$$

Another Form of Beta function:

$$\beta(p,q) = \int_0^{\infty}\frac{y^{q-1}}{(1+y)^{p+q}}dy = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx = \int_0^{\infty}\frac{y^{p-1}}{(1+y)^{p+q}}dy$$

- $\beta(m,n) = \beta(m+1,n)+\beta(m,n+1)$
- Legendre's Duplication formula: $\sqrt{\pi}\ \Gamma(2n) = 2^{2n-1}\Gamma(n)\ \Gamma(n+\frac{1}{2})$
- $\Gamma(n)\ \Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$
- $\displaystyle\int_0^{\infty} e^{-kx}x^{p-1}dx$ where k >0 $= \dfrac{\Gamma(p)}{k^n}$

Prove the following:

1) $\Gamma(n) = \displaystyle\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy$ **2)** $\displaystyle\int_0^{\infty}\frac{x^c}{c^x}dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$

3) $\displaystyle\int_0^{\infty} a^{-bx^2}dx = \frac{\sqrt{\pi}}{2\sqrt{b\log a}}$ **4)** β(m,1/2) = 2^{2m-1}β(m,m)

5) $\displaystyle\int_0^{\pi/2}\sqrt{\tan\theta}\ d\theta = \int_0^{\pi/2}\sqrt{\cot\theta}\ d\theta = \frac{1}{2}\Gamma\left(\tfrac{1}{4}\right)\Gamma\left(\tfrac{3}{4}\right)$ = Π/√2

6) Find $\int_0^{\pi/2}\left(\sqrt{\tan\theta}+\sqrt{\sec\theta}\right)d\theta$

7) Express $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}dx$ as a beta function

8) Prove that $\int_a^b (x-a)^m (b-x)^n\,dx$ = (b-a)$^{\mathbf{m+n+1}}$β(m+1,n+1)

9) Prove that $\int_0^1 x^m(\log x)^n\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$ where n is a positive integer and m >-1

10) Prove that $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1}dy = \frac{\Gamma(p)}{q^p}$ where q > 0 p >0

11) Express the following integrals in terms of gamma functions:

a) $\int_0^1 \frac{1}{\sqrt{1-x^n}}$ **b)** $\int_0^\infty x^n e^{-a^2x^2}\,dx$ **c)** $\int_1^\infty \frac{1}{x^{p+1}+(x-1)^q}dx$ (-p<q<1)

12) Prove that β(n,n)= $\frac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\Gamma(n+1/2)}$

14) Prove that $\int_0^\infty e^{-x^{1/m}}dx$ = m Γ(m)

15) Show that $\int_0^\infty \sqrt{x}e^{-x^3}dx = \frac{\sqrt{\pi}}{3}$

16) Show that $\int_0^{\pi/2}\sin^2\theta\cos^4\theta$ = Π/32

17) Prove that $\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}} = \frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{2}\right)$

18) Show that the area in the first quadrant enclosed by the the curve $\left(\frac{x}{a}\right)^\alpha+\left(\frac{y}{b}\right)^\beta$ =1

where α >0 β>0 is given by $\frac{ab}{\beta+\alpha}\frac{\Gamma(1/\alpha)\Gamma(1/\beta)}{\Gamma(\frac{1}{\alpha}+\frac{1}{\beta})} = \frac{ab}{\alpha+\beta}\beta(\frac{1}{\alpha}+\frac{1}{\beta})$

19) Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of gamma function and evaluate

$$\int_0^1 x^5(1-x^3)^{10}\,dx$$

20) Show that $\Gamma(1/2) = \sqrt{\Pi}/2$

21) Establish a relation between beta and gamma functions

22) State and prove legendre's duplication formula for gamma functions

23) Show that $\beta(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dy = \int_0^1 \frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}dx = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}dy$

24) Prove that $\int_0^{\pi/2} \sin^p x\cos^q x\,dx = ½\ \beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$ and

UNIT - V

MULTIVARIABLE CALCULUS

Functions of Several Variable

A Symbol 'Z' which has a definite value for every pair of values of x and y is called a function of two independent variables x and y and we write Z = f(x,y).

Limit of a Function f(x,y):-

The function f(x,y) defined in a Region R, is said to tend to the limit 'ι' as $x \to a$ and $y \to b$ iff corresponding to a positive number $\in$, There exists another positive number δ such that

$| f(x,y) - \iota | < \in$ for $0 < (x-a)^2 + (y-b)^2 < \delta^2$ for every point (x,y) in R.

Continuity:-

A function f(x,y) is said to be continuous at the point (a,b) if

$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) = f(a,b).$$

Homogeneous Function:-

An expression of the form,

$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + - - - - + a_n y^n$ in which every term is of n^{th} degree, is called a homogeneous function of order 'n'.

Euler's Theorem:-

If $z = f(x,y)$ be a homogeneous function of order 'n' in x and y, then $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$

Functional Dependence and Independence:

Jacobians:-

Definition

The **Jacobian** of the transformation $x = g(u,v)$, $y = h(u,v)$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The Jacobian is defined as a determinant of a 2x2 matrix, if you are unfamiliar with this that is okay. Here is how to compute the determinant.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Therefore, another formula for the determinant is,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Maxima & Minima of functions of two variables:-

A function f(x,y) is said to have a maximum or minimum at x = a, y = b according as f(a+h, b+k) < f(a,b) or f(a+h, b+k) > f(a,b) for all positive or negative small values of h and k.

Procedure to find extreme values of f(x,y):-

Step1:- Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Let $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$

Solving these equations in x & y find the pair of values (a,b) known as stationary points.

Step2:- Calculate r,s,t for each stationary points where $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

Step3:-

1) when $rt - s^2 > 0$

$\rightarrow$ $r < 0$, f(a,b) is maximum

$\rightarrow$ $r > 0$, f(a,b) is minimum

2) when $rt - s^2 < 0$

$\rightarrow$ f(a,b) is not an extreme value

i.e., (a,b) is a saddle point.

3) when $rt - s^2 < 0$, the case is doubtful and needs further investigation.

Method of Lagrange Multipliers

1. Solve the following system of equations.

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$
$$g(x, y, z) = k$$

2. Plug in all solutions, (x, y, z), from the first step into $f(x, y, z)$ and identify the minimum and maximum values, provided they exist.

The constant, λ, is called the **Lagrange Multiplier**.

past finding the point.

Example : Find the maximum and minimum of $f(x,y)=5x-3y$ subject to the constraint $x^2+y^2=136$.

$$5=2\lambda x$$
$$-3=2\lambda y$$
$$x^2+y^2=136$$

$$x=\frac{5}{2\lambda} \qquad y=-\frac{3}{2\lambda}$$

Plugging these into the constraint gives,

$$\frac{25}{4\lambda^2}+\frac{9}{4\lambda^2}=\frac{17}{2\lambda^2}=136$$

We can solve this for λ.

$$\lambda^2=\frac{1}{16} \quad \Rightarrow \quad \lambda=\pm\frac{1}{4}$$

Now, that we know λ we can find the points that will be potential maximums and/or minimums.

If $\lambda=-\frac{1}{4}$ we get,

$$x = -10 \qquad\qquad y = 6$$

and if $\lambda = \dfrac{1}{4}$ we get,

$$x = 10 \qquad\qquad y = -6$$

To determine if we have maximums or minimums we just need to plug these into the function. Also recall from the discussion at the start of this solution that we know these will be the minimum and maximums because the Extreme Value Theorem tells us that minimums and maximums will exist for this problem.

Here are the minimum and maximum values of the function.

$$f(-10,6) = -68 \qquad \text{Minimum at } (-10,6)$$
$$f(10,-6) = 68 \qquad \text{Maximum at } (10,-6)$$

Questions for exercise

1. State Rolle's theorem
2. State Lagranges Mean value theorem.
3. State Cauchy Mean value theorem.
4. State Generalized Mean value theorems.
5. Verify the Rolle's theorem for the function $f(x)=(x-a)^m(x-b)^n$ in [a,b]

   Verify Rolle's theorem for following fuctions:

6. $f(x) = \dfrac{\sin x}{e^x}$ in $[0, \pi]$
7. $f(x) = e^x(\sin x - \cos x)$ in $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$
8. $f(x) = x(x+3)e^{-x/2}$ in $[-3,0]$

9. $f(x) = x^2 - 2x$ in [0,2]

10. $f(x) = |x|$ in [-1,1]

11. $f(x) = \log \dfrac{x^2 + ab}{x(a+b)}$ in [a,b] a>0, b>0

Verify the Lagrange's Mean Value theorem for the following functions

12. $f(x) = \log x$ in [1,e]

13. $f(x) = \begin{cases} \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \quad in \quad [-1,1]$

14.f(x) = x-x^3 in [-2,1]

15 f(x) = $l\,x^2$+mx+n in [a,b]

16 f(x) = x^3-2x^2 in [2,5]

17 f(x) = (x-1)(x-2)(x-3) in [0,4]

Using mean value theorems show that

18. $\dfrac{\pi}{4} + \dfrac{3}{25} < Tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$

19. $\dfrac{\pi}{3} - \dfrac{1}{5\sqrt{3}} > Cos^{-1}\dfrac{3}{5} > \dfrac{\pi}{3} - \dfrac{1}{8}$

20. $\dfrac{x}{1+x} < \log(1+x) < x \qquad \forall \quad x > 0$

Verify Cauchy's mean value theorem for the following:

42 f(x)= $\sqrt{x}$ and g(x)= $\dfrac{1}{\sqrt{x}}$ in [a,b]

22. f(x)= $\dfrac{x^3}{4} - 4x$ and g(x) = x^2 in [0,3]

23. f(x)=x^2 and g(x) =x in [a,b]

24. f(x)= e^x and g(x)= e^{-x} in [a,b]

25. f(x)= sin x and g(x)= cos x in $[0, \frac{\pi}{2}]$

26. f(x)= $\frac{1}{x^2}$ *and* g(x)= $\frac{1}{x}$ in [a,b]

27. Explain about Algebraic and Geometrical interpretation of Rolle's theorem

28. Explain about Geometrical interpretation of Lagrange's theorem

29. If y=log (x+ $\sqrt{1+x^2}$) then find the expansion of y in ascending power's of x

30. Expand f(x) = $2x^3+ 7x^2+ x-6$ in powers of (x-2)

Objective Bits

1. The value of 'C' is Rolle's theorem for f(x) =x^2 in [-1,1] is [ ]

$(a)\ 0$ $(b)\ \frac{1}{2}$ $(c)\ \frac{1}{4}$ $(d)\ -\frac{1}{2}$

2. The value of C in Rolle's theorem for f(x) = Sinx in $[0,\pi]$ is [ ]

$(a)\ 0$ $(b)\ \frac{\pi}{4}$ $(c)\ \frac{\pi}{2}$ $(d)\ \frac{\pi}{6}$

3. The value of C in Rolle's theorem for $f(x)=\frac{\sin x}{e^x} in [0,\pi]$ is [ ]

$(a)\frac{\pi}{4}$

$(b)\frac{\pi}{2}$

$(c)\frac{\pi}{3}$

$(d)0$

4. The value of C in Roller's theorem for $f(x)=\cos\frac{1}{x} in -1\le x\le 1$ is [ ]

$(a)0$

$(b)-\frac{1}{2}$

$(c)\frac{1}{2}$

(d) *Not applicable*

5. The value of C of the Lagrange's mean value theorem for $f(x) = \log x \, in [1, e] is$ [ ]

$(a) e-1$

$(b) e+1$

$(c) \frac{e-1}{2}$

$(d) None$

6. The value of C of the Lagrange's mean value theorem for $f(x) = x^2 - 3x + 2$ in [0,1] is [ ]

$(a) \frac{1}{3}$ $(b) \frac{1}{2}$ $(c) 0$ $(d) -\frac{1}{2}$

7. The value of C of the Lagrange's mean value theorem for $f(x) = x(x-1)(x-2)$ in $\left[0, \frac{1}{2}\right]$ [ ]

$(a) 1 + \frac{\sqrt{21}}{6}$ $(b) 1 - \frac{\sqrt{21}}{6}$ $(c) 0$ $(d) \frac{\sqrt{21}}{6}$

8. Lagrange's mean value theorem f(x)=Secx in $[0, 2\pi]$ is [ ]

(a) Applicable (b) Not Applicable due to non-differentiability

(c)Applicable and $c = \frac{\pi}{2}$ (d) Not Applicable due to discontinuity

9. The value of C in cauchy's mean value theorem for $f(x) = e^x and\ g(x) = e^{-x}$ in $[3, \neg]$ is [ ]

$(a) 4$ $(b) 5$ $(c) \frac{9}{2}$ $(d) 6$

10. The value of C in Cauchy's mean-value theorem for $f(x) = \sqrt{x}\ and\ g(x) = \frac{1}{\sqrt{x}} in [1, 4]$ is[ ]

$(a) \frac{3}{2}$ $(b) \sqrt{2}$ $(c) 2$ $(d) \frac{5}{2}$

11. The value of C in cauchy's mean-value theorem for $f(x)=x^2; g(x)=x^4 in [a,b] is$ [ ]

$(a)\frac{a^2+b^2}{2}$ $(b)\frac{a+b}{2}$ $(c)\frac{\sqrt{a^2+b^2}}{2}$ $(d)\frac{a+b}{4}$

12. The value of C in cauchy's mean – value theorem for $f(x)=\sin x, g(x)=\cos x \ in\left[-\frac{\pi}{2},0\right]$ [ ]

$(a)-\frac{\pi}{4}$ $(b)\frac{\pi}{4}$ $(c)\frac{-\pi}{3}$ $(d)\frac{-\pi}{6}$

13. The expansion of Log (1+x) in Maclaurins series is [ ]

$(a)x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+--$ $(b)x+\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+--$

$(c)x+\frac{x^2}{2}+\frac{x^3}{3!}-\frac{x^4}{4!}+--$ $(d)x-\frac{x^2}{2}+\frac{x^3}{3!}-\frac{x^4}{4!}+--$

14. The Maclaurins series for Cosx is [ ]

$(a)1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6}+--$ $(b)1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+--$

$(c)x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+--$ $(d)x+\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+--$

15. $f(a+b)=f(a)+hf^1(a)+\frac{h^2}{2!}f^{11}(a)+----+\frac{h^n}{n!}+n(a+\theta h)$ is called [ ]

(a) Taylor's theorem with lagrange form of remainder
(b) Cauchy's theorem with Lagrange form of remainder
(c)Lagrange's theorem with Lagrange form of remainder
(d) Maclaurin's theorem with Lagrange form of remainder

16. If $U=x^y then \frac{\partial u}{\partial x}=$ [ ]

$(a)\ yx^{y-1}$ $(b)\ x^y \log x$ $(c)\ 0$ $(d)\ x^y$

17. If $U=x^y then \frac{\partial u}{\partial y}=$ [ ]

$(a)\ yx^{y-1}$ $(b)\ x^y$ $(c)\ x^y \log x$ $(d)\ 0$

18. The degree of homogeneous function $\frac{\sqrt{x}+\sqrt{y}}{x+y} is$ [ ]

$(a)\frac{1}{2}$ $(b)-\frac{1}{2}$ $(c)1$ $(d)0$

19. If $u=\sin^{-1}\frac{x}{y}$ then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=$ [ ]

$(a)1$ $(b)0$ $(c)4$ $(d)-1$

20. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$ [ ]

(a) sineu (b) Cos2u (c) –sin2u (d) 0

21. If $u = xy^2\phi\left(\frac{x}{y}\right)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} =$ [ ]

(a) 2u (b) 3u (c) u (d)0

22. If $u = x^2y^2, x = r\cos\theta, y = r\sin\theta$ then $\frac{du}{dr} =$ [ ]

$(a)\, xy(x\cos\theta + y\sin\theta)$ $(b)\, 2xy(y\cos\theta + x\sin\theta)$ /

$(c)\, y(x\sin\theta + y\cos\theta)$ $(d)\, xy(\sin\theta + \cos\theta)$

23.If $u = x+y+z; x = t, y = t^2, z = t^3$ then $\frac{du}{dt} =$ [ ]

$(a)\, t+t^2+t^3$ $(b)\, 1+2t+t^2$ $(c)\, 1+2t+3t^2$ $(d)\, 1+2t-3t^2$

24. If $u = x^2+y^2+z^2, x = e^t, y = e^t\sin t, z = e^t\cos t$ then $\frac{du}{dt} =$ [ ]

$(a)\, 2e^{2t}$ $(b)\, 3e^{2t}$ $(c)\, 4e^{2t}$ $(d)\, 0$

25. If $u = f(r)$ and $r = x + y^2$ then $\frac{\partial^2 u}{\partial y^2} =$ [ ]

$(a)\, 2f^1(r)+4y^2f^{11}(r)$ $(b)\, f^1(r)+2yf^{11}(r)$ $(c)\, f^1(r)+yf^{11}(r)$ $(d)\, f^1(r)-yf^{11}(r)$

26. If $x = r\cos\theta, y = r\sin\theta$ then $\frac{\partial(x,y)}{\partial(r,\theta)} =$ [ ]

$(a)\, \cos\theta$ $(b)\, \sin\theta$ $(c)\, r$ $(d)\, \frac{1}{r}$

27. If $x = r\cos\theta, y = r\sin\theta$ then $\frac{\partial(r,\theta)}{\partial(x,y)} =$ [ ]

$(a)\, 0$ $(b)\, 1$ $(c)\, r$ $(d)\, \frac{1}{r}$

28. If $u = \frac{x+y}{1-xy}; r = \tan^{-1}x + \tan^{-1}y$ then $J\left(\frac{u,v}{x,y}\right).J\left(\frac{x,y}{u,v}\right) =$ [ ]

$(a)\, 0$ $(b)\, 1$ $(c)\, -1$ $(d)\, None$

29. Two functions u,v are said to be functionally dependent if $J\left(\frac{u,v}{x,y}\right)$ is [ ]

(a) Not equal to zero (b) Negative (c) Equal to zero (d) None

30. A function f(x,y) will have a maximum value at (a,b) if – where $r = f_{xx}; s = f_{yy}$ [ ]

$(a)\, rt-s^2<0$ and $r<0$ $(b)\, rt-s^2>0$ and $r<0$ $(c)\, rt-s^2>0$ and $r>0$ $(d)\, rt-s^2=0$